# Term Structure of Risk under Alternative Econometric Specifications*

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May 1, 2004

**Abstract**

This paper characterizes the term structure of risk measures such as Value at Risk (VaR) and expected shortfall under different econometric approaches including multivariate regime switching, GARCH-in-mean models with student-t errors, two-component GARCH models and a non-parametric bootstrap. We show how to derive the risk measures for each of these models and document large variations in term structures across econometric specifications. An out-of-sample forecasting experiment applied to stock, bond and cash portfolios suggests that the best model is asset- and horizon specific but that the bootstrap and regime switching model are best overall for VaR levels of 5% and 1%, respectively.

Key words: term structure of risk, nonlinear econometric models, simulation methods.

## 1. **Introduction**

Quantitative models are now routinely used in risk management and have been the subject of extensive academic interest as witnessed by the many survey papers and monographs on the topic, c.f. Duffie and Pan (1997), Manganelli and Engle (2001) and Christoffersen (2003) and papers such as Britten-Jones and Schaeffer (1999) and Berkowitz and O'Brien (2002). The early literature was mainly built around volatility models, but subsequent studies have progressed to study risk

*We thank three anonymous referees and the editor, Frank Diebold, for many comments and suggestions that helped to significantly improve the paper.

measures such as Value at Risk (VaR) and expected shortfall.[1] Value at Risk is used to monitor risk exposure for regulatory purposes and also to gauge risk adjusted investment performance.

In a typical portfolio optimization exercise the investor attempts to maximize the expected value of some objective function subject to a constraint requiring that the (conditional) risk measure does not exceed a pre-specified level. The risk measure is often calculated under a maintained econometric model so it is important to use a model that is not misspecified. For example, if risk is underestimated, there is a higher than expected chance of incurring large losses and regulatory fines may ensue. Conversely, underestimation of risk leads investors to maintain needlessly large reserves with an associated increase in their cost of capital.

In this paper we consider the term structures of commonly used risk measures under a range of econometric specifications including multivariate regime switching, multivariate GARCH-in-mean models with fat tails, and two-component GARCH models fitted to univariate portfolio return series. These are highly nonlinear dynamic specifications that account for time-varying mean, variance and higher order moments, so we propose to use simulation methods to explore their term structure implications at several horizons. We also study risk under a non-parametric bootstrap.

While most studies on risk modeling have focused on relatively short horizons using high frequency data, as argued by Christoffersen, Diebold and Schuerman (1998), the relevant horizon can be very long and depends on the economic problem at hand. Our application considers investors' strategic asset allocation and studies portfolios composed of broadly defined asset classes such as T-bills (cash), bonds and stocks. We find evidence of large variations both in levels and shapes of term structures of risk measures across econometric specifications.

The contribution of the paper is three-fold. First, we provide econometric estimates for a range of econometric models for returns on stock and bond portfolios, using both univariate and multivariate specifications. Second, we characterize the term structure of risk measures such as VaR and expected shortfall under each of the econometric models. Despite the popularity of the included models, to our knowledge the term structure of risk of such models has not previously been subject to a comparative study such as ours. Third, we analyze the predictive performance of the econometric models in an out-of-sample experiment. Our results suggest that the best approach may be asset- and horizon specific and also depends on how far out in the tails risk is measured.

The structure of the paper is as follows. Section 2 introduces the econometric specifications

[1] Diebold, Gunther and Tay (1998) go one step further and consider models for the entire predictive density.

and provides empirical estimates. Section 3 characterizes the term structure of VaR and expected shortfall under the univariate and multivariate models from Section 2. Section 4 provides results from an out-of-sample forecasting experiment using model-free diagnostic tests. Section 5 concludes.

## 2. **Econometric Specifications**

We consider models for an $n \times 1$ vector of asset returns measured in excess of the T-bill rate $r^f$, $\mathbf{r}_t = (r_{1t}, ...., r_{nt})'$. At the most basic level, one has to decide whether to model the multivariate return dynamics for $\mathbf{r}_t$ or $-$ if portfolio weights, $\boldsymbol{\omega}$, are fixed $-$ simply study the univariate time series of portfolio returns, $R_t$. Univariate models include two-component GARCH specifications fitted directly to portfolio return series (Engle and Lee (1999)), while multivariate models include GARCH models (Engle and Kroner (1995)) possibly extended to include time-varying means and fat-tailed (e.g., student-t) innovations, and regime switching models (e.g. Ang and Bekaert (2002) and Guidolin and Timmermann (2003)). These models are designed to capture persistence in volatility as well as fat tails (outliers), c.f. Chernov, Gallant, Ghysels and Tauchen (2003). The models also differ in how they accommodate time-variations in expected means (e.g., through GARCH-in-mean effects or through regime-dependent intercepts). As argued by Christoffersen (2003), this can have sizeable consequences for risk measures, particularly at long horizons.

In the following we describe the econometric specifications considered in our study. A natural multivariate benchmark model is the Gaussian specification

$$\mathbf{r}_t = \boldsymbol{\mu} + \sum_{j=1}^{p} \mathbf{A}_j \mathbf{r}_{t-j} + \boldsymbol{\psi}\boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t = (\varepsilon_{1,t}, ..., \varepsilon_{n,t})' \sim IIN(\mathbf{0}, \mathbf{I}_n). \tag{1}$$

$\boldsymbol{\psi}$ is the Cholesky factor of the covariance matrix, i.e. $E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] = \boldsymbol{\psi}'\boldsymbol{\psi} = \Omega$.

Multivariate regime switching models let the mean, covariance and any serial correlations in returns be driven by a common state variable, $S_t$, that takes integer values between 1 and $k$:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \mathbf{r}_{t-j} + \boldsymbol{\psi}_{s_t} \boldsymbol{\varepsilon}_t. \tag{2}$$

Here $\boldsymbol{\mu}_{s_t} = (\mu_{1s_t}, ...., \mu_{ns_t})'$ is an $n \times 1$ vector of mean returns in state $s_t$, $\mathbf{A}_{j,s_t}$ is the $n \times n$ matrix of autoregressive coefficients associated with lag $j \geq 1$ in state $s_t$, and $\boldsymbol{\varepsilon}_t \sim IIN(\mathbf{0}, \mathbf{I}_n)$ follows a multivariate normal distribution with zero mean. $\boldsymbol{\psi}_{s_t}$ is now a state-dependent Choleski factorization of the covariance matrix $\Omega_{s_t}$, i.e. $\boldsymbol{\psi}'_{s_t} \boldsymbol{\psi}_{s_t} = \Omega_{s_t}$. Regime switches in the state variable,

$S_t$, are assumed to be governed by the transition probability matrix, $\mathbf{P}$, with elements

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i, j = 1, .., k. \tag{3}$$

Each regime is thus the realization of a first-order Markov chain with constant transition probabilities. This model is quite general and lets asset returns have different means, variances and correlations in different states, allowing risk measures to vary considerably across states.

Our third multivariate specification is a constant correlation GARCH(1,1)-M model:

$$\begin{aligned} r_{i,t} &= \xi_i + \sum_{j=1}^{n} \sum_{k \geq j}^{n} \lambda_{i,jk} \sigma_{jk,t}^2 + \eta_{i,t}, \\ \sigma_{ii,t}^2 &= \alpha_{ii} + \beta_{i0} \eta_{i,t-1}^2 + \beta_{i1} \sigma_{ii,t-1}^2, \quad i = 1, ..., n \\ \sigma_{ij,t}^2 &= \rho_{ij} \sigma_{ii,t-1} \sigma_{jj,t-1}, \quad i, j = 1, ..., n, \quad i \neq j. \end{aligned} \tag{4}$$

Here $\boldsymbol{\eta}_t = (\eta_{1,t}, \eta_{2,t}, .., \eta_{n,t})'$ are heteroskedastic return innovations defined as $\boldsymbol{\eta}_t = \boldsymbol{\psi}_t \boldsymbol{\varepsilon}_t$, and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, ..., \varepsilon_{n,t})' \sim IID(\mathbf{0}, \mathbf{I}_n)$. $\boldsymbol{\psi}_t' \boldsymbol{\psi}_t = \Omega_t = [\sigma_{ij,t}^2]$ is the conditional covariance matrix and $\rho_{ij}$ is the correlation coefficient which is assumed to be constant, c.f. Bollerslev (1990). This model captures both first and second moment dynamics. As established by the recent empirical finance literature, volatility models generally need two components, one to capture persistence in volatility and another one to generate fat tails. While a regime switching model generates persistent volatility and fat tails through the switching mechanism, a Gaussian GARCH(1,1)-M model might be too restrictive to produce tails that are fat enough. To capture fat tails we therefore consider student-t innovations.[2]

$$f(\boldsymbol{\varepsilon}_t; v) = \frac{\Gamma\left((v+2)/2\right)}{(\pi v)\Gamma\left(v/2\right)} \left(1 + \frac{1}{v} \boldsymbol{\varepsilon}_t' \boldsymbol{\varepsilon}_t\right)^{-\frac{v+2}{2}} \qquad v > 2. \tag{5}$$

Our final parametric model applies to the univariate dynamics of portfolio returns $R_t$ using the two component GARCH(1,1) model proposed by Engle and Lee (1999), where a second component governs changes to the long-run volatility and accommodates thick tails. Moreover, the model is naturally extended to accommodate leverage effects allowing negative portfolio return shocks to have a larger impact on volatility than positive shocks, c.f. Glosten et al. (1993):

$$\begin{aligned} R_t &= \xi + \lambda \sigma_t^2 + \eta_t \\ \sigma_t^2 &= q_t + \beta_0 \left(\eta_{t-1}^2 - q_{t-1}\right) + \beta_1 \left(\sigma_{t-1}^2 - q_{t-1}\right) + \beta_2 \left(D_{t-1} \eta_{t-1}^2 - \frac{1}{2} q_{t-1}\right), \\ q_t &= \omega + \rho q_{t-1} + \phi \left(\eta_{t-1}^2 - q_{t-1}\right). \end{aligned} \tag{6}$$

[2] $\Gamma(\cdot)$ is the Gamma function and $v$ is the degree of freedom parameter. We thank an anonymous referee for pointing out the suitability of a fat-tailed bivariate GARCH for benchmarking in our application.

Here $\eta_t = \sigma_t \varepsilon_t$, and $\varepsilon_t \sim IIN(0,1)$, $D_t = 1$ if $\eta_t < 0$ and $D_t = 0$ if $\eta_t \geq 0$. Positive innovations have a marginal effect of $\beta_0$ on the conditional volatility, while negative innovations have an effect of $\beta_0 + \beta_2$.[3] $q_t$ is the permanent volatility component, while departures from the permanent component, $\eta_{t-1}^2$ and $\sigma_{t-1}^2$, determine short-run volatility. Since no multivariate generalizations of (6) have been proposed, we estimate this model directly to the univariate portfolio return series, $R_t$.

We also consider risk measures computed using a nonparametric block bootstrap. This is done by resampling the sample indices $\{1, 2, ..., T\}$ where $T$ is the sample size and the block length $L$ is set equal to the forecast/investment horizon, $h$. Risk measures are then computed as averages across $Q$ bootstrap resamples.

### 2.1. *Data*

Our application studies portfolios comprising three major US asset classes, namely stocks, bonds and T-bills. Decisions on how much to invest in such broadly defined asset classes are commonly referred to as strategic asset allocation and have recently been the subject of considerable interest (e.g. Brennan, Schwarz and Lagnado (1997) and Campbell and Viceira (2001)). Our focus on a small set of asset classes has the further advantage that it poses a manageable problem for the econometric analysis and for computations of term structures of risk at several horizons.

Our analysis uses monthly returns on the value-weighted portfolio of all common stocks listed on the NYSE. We also consider the return on a portfolio of 10-year Treasury bonds. Returns are calculated applying the standard continuous compounding formula, $\tilde{r}_t \equiv \ln V_t - \ln V_{t-1}$, where $V_t$ is the asset price, inclusive of any cash distributions (dividends, coupons) between time $t-1$ and $t$. To obtain excess returns, $r_t$, we subtract the 30-day T-bill rate ($r_t^f$) from these returns, $r_t \equiv \tilde{r}_t - r_t^f$. All data is obtained from the Center for Research in Security Prices. Hence the multivariate models focus on the bivariate dynamics in stock and bond excess returns, taking the risk-free rate as given. Our sample is January 1954 - December 1999, a total of 552 observations.

### 2.2. *Parameter Estimates*

To determine the design parameters of the regime switching model (2), we studied a range of bivariate specifications with different numbers of states, $k$, and lag order, $p$. We found no evidence of any

[3] Leverage effects might also appear in the permanent component, but Engle and Lee (1999) and others have generally found that they are weaker than those in the transitory component.

significant autoregressive terms but strongly rejected the null of a single state using Davies' (1977) upper bound for the critical values. The Hannan-Quinn and BIC information criteria supported a four-state specification without lagged returns ($k = 4$, $p = 0$).[4]

The parameter estimates for the four-state regime switching model fitted to stock and bond excess returns, $\mathbf{r}_{t+1} = (r_{t+1}^{stock}\ r_{t+1}^{bond})'$, were as follows (with one, two and three stars indicating significance at the 10%, 5% and 1% critical levels, respectively):[5]

$$\text{State 1:} \quad \mathbf{r}_{t+1} = \begin{bmatrix} \underset{(0.0202)}{-0.0845} \\ \underset{(0.0078)}{-0.0015} \end{bmatrix} + \boldsymbol{\varepsilon}_{t+1} \quad \hat{\Omega}_1 = \begin{bmatrix} 0.0539^{***} & \cdot \\ -0.8513^{***} & 0.0242^{***} \end{bmatrix}$$

$$\text{State 2:} \quad \mathbf{r}_{t+1} = \begin{bmatrix} \underset{(0.0021)}{0.0091} \\ \underset{(0.0009)}{-0.0003} \end{bmatrix} + \boldsymbol{\varepsilon}_{t+1} \quad \hat{\Omega}_2 = \begin{bmatrix} 0.0359^{***} & \cdot \\ 0.2008^{**} & 0.0164^{***} \end{bmatrix}$$

$$\text{State 3:} \quad \mathbf{r}_{t+1} = \begin{bmatrix} \underset{(0.0046)}{0.0126} \\ \underset{(0.0006)}{0.0001} \end{bmatrix} + \boldsymbol{\varepsilon}_{t+1} \quad \hat{\Omega}_3 = \begin{bmatrix} 0.0289^{***} & \cdot \\ -0.0288 & 0.0032^{**} \end{bmatrix}$$

$$\text{State 4:} \quad \mathbf{r}_{t+1} = \begin{bmatrix} \underset{(0.0047)}{0.0099} \\ \underset{(0.0032)}{-0.0044} \end{bmatrix} + \boldsymbol{\varepsilon}_{t+1} \quad \hat{\Omega}_4 = \begin{bmatrix} 0.0479^{***} & \cdot \\ 0.4431^{***} & 0.0336^{***} \end{bmatrix}$$

$$\hat{\mathbf{P}} = \begin{pmatrix} 0.4940^{**} & 0.0215 & 0.0605^{*} & 0.4239^{**} \\ 0.0181 & 0.9767^{***} & 0.0000 & 0.0053 \\ 0.0000 & 0.0266 & 0.9734^{***} & 0.0000 \\ 0.0148 & 0.0563^{**} & 0.0000 & 0.9290^{***} \end{pmatrix}.$$

There are significant time-variations in the first and second moments of the joint distribution of stock and bond returns across the four regimes. Mean excess returns on stocks vary from 1.3% per month - almost double their unconditional mean - in the third state to minus 8.5% per month in

[4]Both criteria have been used in the literature on nonlinear econometric modeling, c.f. Sin and White (1996). As shown by Hannan (1987), the Hannan-Quinn criterion is strongly consistent. Following Kilian and Ivanov (2001) we also adopted the Akaike criterion which selected a very large model with sixty parameters and regimes with very small ergodic state probabilities.

[5]Estimates on the diagonal of $\Omega$ are monthly standard deviations, while off-diagonal estimates are correlation coefficients. The MLE estimates for the IID Gaussian model are 0.0067 and 0.0008 for the means, 0.0424 and 0.0224 for the volatilities, and 0.2386 for the correlation coefficient.

the first state. Interestingly, all four estimates of mean stock returns are statistically significant at conventional levels. Less variation is observed in the mean returns of long-term bonds and none of the mean estimates is significant for this asset class.

Turning to the volatility and correlation parameters, stock return volatility varies between 2.9% and almost 5.4% per month, with state one displaying the highest value. Bond return volatility varies even more across states, going from a very low value of 0.3% per month in state three to 3.4% per month in state four. Correlations between stock and bond returns go from -0.85 in state one to 0.44 in state four. In state three stock and bond returns are essentially uncorrelated. Such large differences in correlations across states are important for risk management purposes and support using a multi-state model.

The transition probability estimates and plots of the smoothed state probabilities revealed that state one is a transitory state with relatively isolated spikes lasting on average around two months and capturing high volatility episodes that occur in only 3% of the sample. States two and four, on the other hand are highly persistent and capture 67% and 23% of the sample, respectively. State three largely identifies a single historical episode from 1962-1966, although its ergodic probability is 7%. The fact that their overall ergodic probabilities are only 10% does not mean that states one and three are unimportant for risk management and modeling purposes, however. Clearly the left tail of the distribution of stock and bond returns is significantly affected particularly by state one and the possibility of shifting to this state even if starting from another state.[6]

Turning to the GARCH(1,1)-M specification (4), we obtained the following parameter estimates by MLE:

$$\mathbf{r}_t = \begin{bmatrix} \underset{(0.0091)}{0.1378} \\ \underset{(0.0025)}{-0.0192} \end{bmatrix} + \begin{bmatrix} \underset{(0.0265)}{1.4846} & \underset{(0.0352)}{-87.4217} & \underset{(0.2053)}{-250.6173} \\ \underset{(0.0241)}{-2.3630} & \underset{(0.2067)}{104.5330} & \underset{(0.2351)}{0.0012} \end{bmatrix} \begin{bmatrix} \sigma^2_{11,t} \\ \sigma^2_{12,t} \\ \sigma^2_{22,t} \end{bmatrix} + \boldsymbol{\eta}_t,$$

$$\hat{\Omega}_t = \begin{bmatrix} \underset{(0.0054)}{0.0011} + \underset{(0.0082)}{0.0502}\eta^2_{1,t-1} + \underset{(0.0582)}{0.4881}\sigma^2_{11,t-1} & \cdot \\ \underset{(0.0221)}{0.2525} \times \sigma_{11,t-1} \times \sigma_{22,t-1} & \underset{(0.0001)}{0.0005} + \underset{(0.0402)}{0.0140}\eta^2_{2,t-1} + \underset{(0.0390)}{0.4595}\sigma^2_{22,t-1} \end{bmatrix} \quad \hat{v} = \underset{(0.0208)}{6.8721}.$$

The implied persistence of volatility is moderate for both asset classes. Stock return volatility

[6] Guidolin and Timmermann (2003) model the non-linear dynamics of a larger asset menu including size-sorted equity portfolios comprising small and large stocks. They also find evidence that four states are needed to model the dynamics in the joint returns on stock and bond portfolios but stop short of investigating the implications of this for risk measures or predictive performance.

scarcely affects mean returns on either asset class while bond return volatility has a considerable effect on the equity premium. The estimate of $v$ indicates very fat tails. Interestingly, LR tests comparing either a Gaussian GARCH(1,1) without 'in-mean' effects or a Gaussian GARCH(1,1)-M against the more general specification (4) strongly rejected the simpler models so both time-varying mean effects and fat tails appear to be empirically important.

Turning to the univariate GARCH-components models, we used MLE to fit (6) to five portfolios representing different levels of aggressiveness in the asset allocation: (i) 50% stocks and 50% bonds, (ii) 50% bonds and 50% T-bills, (iii) 100% bonds, (iv) 50% stocks and 50% T-bills and (v) 100% stocks. The results were as follows:

| | 50% bonds 50% T-bills | 100% bonds | 50% stocks 50% T-bills | 50% stocks 50% bonds | 100% stocks |
|---|---|---|---|---|---|
| $\xi$ | $-0.087$ (0.0005) | $-.0008$ (.0009) | .0012 (.0018) | .0016 (.0019) | .0024 (.0035) |
| $\lambda$ | 5.6509 (5.6161) | 6.2581 (2.6781) | 5.0245 (4.1948) | 3.9601 (2.9876) | 2.4627 (2.0053) |
| $\beta_0$ | .0055 (0.0400) | .0084 (.0418) | .2747 (.0356) | .1557 (.4180) | .2760 (.0169) |
| $\beta_1$ | .5168 (.3577) | .4896 (.4165) | .5550 (.1424) | .4334 (.1852) | .5464 (.1377) |
| $\beta_2$ | $-.0049$ (.0725) | $-.0024$ (.0736) | .2707 (.0087) | .3178 (.0886) | .2769 (.0556) |
| $\omega$ | .0004 (.0007) | .0011 (.0024) | .0004 (.0001) | .0007 (.0003) | .0018 (.0003) |
| $\rho$ | .9975 ().0020 | .9976 (.0048) | .9347 (.0310) | .9718 (.0183) | .9378 (.0318) |
| $\phi$ | .1538 (.0239) | .1515 (.0256) | .1285 (.0245) | .1058 (.0277) | .1299 (.0184) |

Most of the estimated coefficients in the variance equations are highly significant. Leverage effects are not important for the bond portfolios but appear to be strong for portfolios involving stocks with $\beta_2 > 0$. Bad news therefore have a stronger impact on volatility than good news. Long-run volatility, $q_t$, is quite persistent, especially for bonds where $\hat{\rho}$ is close to 1. Interestingly, we find a significant feedback from volatility to mean excess returns for only one of the five portfolios (100% bonds), although the estimated coefficient is small in economic terms.

## 3. **Term Structures of Risk Measures**

It is important to study how risk measures vary across different horizons. For example, the entire term structure of VaR estimates is relevant even to long-term risk managers who may choose to adjust portfolio weights significantly following changes to short-term VaR estimates. Solutions to the dynamic programming problem faced by a risk manager with a long investment (planning) horizon will also reflect VaR estimates at short or intermediate (decision) horizons provided that portfolio weights can be rebalanced at interim points.

The econometric models fitted in the previous section provide rich dynamic specifications for asset returns. We use these models to study the term structure of risk measures in common use such as Value at Risk and expected shortfall. Let the weight on stocks and bonds be $\boldsymbol{\omega} = (\omega^{stock}, \omega^{bond})'$. Then the cumulated $h-$period gross return on the portfolio comprising stocks, bonds and T-bills from period $t$ to period $t+h$ is

$$R_{t:t+h} = (1 - \boldsymbol{\omega}'\boldsymbol{\iota}_2) \exp\left(hr^f\right) + \boldsymbol{\omega}' \exp\left(\mathbf{r}_{t:t+h} + hr^f\boldsymbol{\iota}_2\right), \tag{7}$$

where $r^f$ is the risk-free rate, $\mathbf{r}_{t:t+h} \equiv \sum_{i=1}^{h} \mathbf{r}_{t+i}$ is the (continuously compounded) $h$-period cumulated excess returns and $\boldsymbol{\iota}_2$ is a $2 \times 1$ vector of ones.[7]

### 3.1. *Value at Risk*

Value at Risk ($VaR$) at the $h$-period horizon is simply the $\alpha$ quantile of the conditional probability distribution of $R_{t:t+h}$. Reporting $VaR$ as a positive number representing a loss, we have

$$\Pr(R_{t:t+h} \leq -VaR^{\alpha}_{t:t+h}|\mathcal{F}_t) = \alpha, \tag{8}$$

where $\mathcal{F}_t$ is the period-$t$ information set. We follow common practice and assume that $\mathcal{F}_t = \{\mathbf{r}_j\}_{j=1}^{t}$ comprises information on past asset returns. For a given asset allocation, $\boldsymbol{\omega}$, the $VaR$ estimate depends on the horizon, $h$, the significance level, $\alpha$, the information set, $\mathcal{F}_t$, and on the econometric model. As portfolio weights are changed, the $VaR$ estimate also changes.

It is useful to establish how the VaR measure evolves as a function of $h$ under the simple homoskedastic Gaussian benchmark for portfolio returns, $R_t \sim IIN(\mu, \sigma^2)$. Since $R_{t:t+h} \sim$

[7]We define the exponential of an $n \times 1$ vector as an $n \times 1$ vector that collects the exponentials of each of the elements of the original vector.

$N(\mu h, \sigma^2 h)$,

$$VaR^{\alpha}_{t:t+h} = -\mu h - \sigma\sqrt{h}\Phi^{-1}(\alpha). \tag{9}$$

Differentiating with respect to $h$, we get the first order condition $\mu + \sigma\Phi^{-1}(\alpha)/2\sqrt{h} = 0$. The horizon where VaR peaks (assuming a positive mean return, $\mu$) is therefore $h^* = \sigma^2(\Phi^{-1}(\alpha))^2/4\mu$. For small values of $h$ VaR may initially rise, but for longer horizons it will eventually be dominated by the linear term, $\mu h$, and will thus decline provided $\mu$ is positive.

### 3.2. *Calculation of VaR*

When returns follow one of the nonlinear specifications from Section 2, there are no closed-form expressions for the VaR. For example, for the regime switching process, the $h$-step ahead distribution of portfolio returns conditional on being in state $s_t$ at time $t$, $P(R_{t:t+h}|S_t = s_t)$ is not Gaussian but rather a mixture of $k$ normal distributions.[8] In practice, the current state is not observable and a risk manager will have to rely on the most recent state probability estimates computing the probability of the time $t+h$ regimes $s_{t+h}$ by $\pi_{s_{t+h}|t} = \boldsymbol{\pi}_t \mathbf{P}^h \mathbf{e}_{s_{t+h}}$ with $\boldsymbol{\pi}_t \equiv [\Pr(s_t = 1|\mathcal{F}_t)\ \Pr(s_t = 2|\mathcal{F}_t)\ \ldots\ \Pr(s_t = k|\mathcal{F}_t)]$ being a row vector of period-$t$ probabilities and $\mathbf{e}_s$ a $k \times 1$ vector with a 1 in the $s-$th position and zeros everywhere else.

We first explain our approach to characterize the term structure of risk in the context of the regime switching model and then show how to adapt the algorithm to the GARCH specifications. For a given set of portfolio weights, $\boldsymbol{\omega}$, under the regime switching model the conditional c.d.f. of the single-period portfolio return given the current information, $\mathcal{F}_t$, can be computed as follows:

$$\begin{aligned}
F(R_{t:t+h}(\boldsymbol{\omega}) \leq \kappa|\mathcal{F}_t) &= \int_{\Lambda(\boldsymbol{\omega},\kappa)} \left\{ \int_{-\infty}^{\lambda_2^*} \int_{-\infty}^{\lambda_1^*} \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}|t} \right. \\
&\quad \left. \times\ \phi_2\left(\Psi_{s_{t+h}}^{-1}\left[\mathbf{r}_{t+h} - \boldsymbol{\mu}_{s_{t+h}}\right]\right) dr_{1,t+h} dr_{2,t+h} \right\} \cdot d\boldsymbol{\lambda}^* \\
&= \int_{\Lambda(\boldsymbol{\omega},\kappa)} \left\{ \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}|t} \int_{-\infty}^{\lambda_2^*} \int_{-\infty}^{\lambda_1^*} \phi_2(\mathbf{x}) dx_1 dx_2 \right\} \cdot d\boldsymbol{\lambda}^* \\
&= \int_{\Lambda(\boldsymbol{\omega},\kappa)} \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}|t} \Phi_2(\lambda_1^*, \lambda_2^*; \boldsymbol{\mu}_{s_{t+h}}, \boldsymbol{\psi}_{s_{t+h}}) \cdot d\boldsymbol{\lambda}^*, \qquad (10)
\end{aligned}$$

[8] Furthermore, (7) shows that gross portfolio returns are weighted sums of the exponential of (linear transforms of) continuously compounded returns.

where $\Phi_2(\cdot;\boldsymbol{\mu}_{s_{t+h}},\boldsymbol{\psi}_{s_{t+h}})$ is the standard bivariate normal c.d.f., $\phi_2(\cdot)$ is the corresponding bivariate p.d.f. both depending on the standardized deviations of $\mathbf{r}_{t+h}$ from its mean vector, i.e. $\Psi_{s_{t+h}}$ is the Choleski factor of the state-dependent covariance matrix such that

$$\Psi_{s_{t+h}}\Psi'_{s_{t+h}} = \boldsymbol{\psi}'_{s_{t+h}}\boldsymbol{\psi}_{s_{t+h}} + \left(\mathbf{M}' - \boldsymbol{\mu}_{s_{t+h}} \otimes \boldsymbol{\iota}'_k\right)\left(\mathbf{M} - \boldsymbol{\mu}'_{s_{t+h}} \otimes \boldsymbol{\iota}_k\right).$$

In this expression $\mathbf{M}$ is defined as (c.f. Timmermann (2000)):

$$\mathbf{M} \equiv [\mu_{is_t}] = \begin{pmatrix} \mu_{11} & \mu_{21} & \cdots & \mu_{n1} \\ \mu_{12} & \mu_{22} & & \mu_{n2} \\ \vdots & \vdots & \ddots & \\ \mu_{1k} & \mu_{2k} & & \mu_{nk} \end{pmatrix}.$$

Finally $\boldsymbol{\lambda}^* \equiv [\lambda_1^*\ \lambda_2^*]' \in \Lambda(\boldsymbol{\omega},\kappa)$ is a $2\times 1$ vector function of $\kappa$ and $\boldsymbol{\omega}$, with the $\Lambda$ region defined as

$$\Lambda(\boldsymbol{\omega},\kappa) \equiv \left\{\boldsymbol{\lambda}^* : (1-\boldsymbol{\omega}'\boldsymbol{\iota}_2)\exp\left(hr^f\right) + \boldsymbol{\omega}'\exp\left(\boldsymbol{\lambda}^* + hr^f\boldsymbol{\iota}_2\right) \le \kappa\right\}.$$

Integrating over the set $\boldsymbol{\lambda}^* \in \Lambda(\boldsymbol{\omega},\kappa)$ is the same as considering all combinations of portfolios with total returns at or below $\kappa$.

While in principle the VaR under regime switching can be computed using (10), in practice it is difficult to compute the integral over the set $\Lambda(\boldsymbol{\omega},\kappa)$, so Monte Carlo simulations appear to be more attractive, using the following algorithm:

1. For each possible value of the current regime, $s_t = 1,...,k$, simulate $Q$ $h-$period (vector) returns $\{\mathbf{r}^q_{t:t+h}(s_t)\}_{q=1}^Q$ from the regime switching model[9]

$$\mathbf{r}_{t+\tau,q}(s_t) = \boldsymbol{\mu}_{s_{t+\tau}} + \boldsymbol{\psi}_{s_{t+\tau,q}}\boldsymbol{\varepsilon}_{t+\tau,q},$$

where $\mathbf{r}^q_{t:t+h}(s_t) \equiv \sum_{\tau=1}^h \mathbf{r}_{t+\tau,q}(s_t)$, $\boldsymbol{\varepsilon}_{t+\tau,q} \sim IIN(\mathbf{0},\mathbf{I}_n)$. This step allows for regime switching at each point in time as governed by the transition matrix, $\mathbf{P}$. For example, starting from state 1 there is a probability $p_{12} \equiv \mathbf{e}'_1\mathbf{P}\mathbf{e}_2$ of switching to regime 2 between period 1 and 2, a probability $p_{11} \equiv \mathbf{e}'_1\mathbf{P}\mathbf{e}_1$ of staying in state 1 and so forth.

[9] $\mathbf{r}^q_{t:t+h}$ denotes cumulative $h-$period vector (excess) returns generated by the $q-$th simulation path, $\sum_{\tau=1}^h \mathbf{r}_{t+\tau,q}$, while $\mathbf{r}_{t+\tau,q}$ denotes a vector of (excess) returns for time $t+\tau$ generated by the $q-$th simulation path. $R^q_{t:t+h}$ is the corresponding $h-$period portfolio return when the weights $\boldsymbol{\omega}$ are fixed. Continuously compounded returns on individual assets are denoted as lowercase $r$ while portfolio returns are in uppercase $R$.

2. Form regime-specific $h-$period portfolio returns using the portfolio weights $\boldsymbol{\omega}$, $R^q_{t:t+h}(\boldsymbol{\omega},s_t) \equiv (1-\boldsymbol{\omega}'\boldsymbol{\iota}_2)\exp\left(hr^f\right) + \boldsymbol{\omega}'\exp(\mathbf{r}^q_{t:t+h}(s_t) + hr^f\boldsymbol{\iota}_2)$.

3. Combine the simulated $h-$period portfolio returns $\{R^q_{t:t+h}(s_t)\}^Q_{q=1}$using the probability weights from the vector $\boldsymbol{\pi}_t$:

$$R^q_{t:t+h}(\boldsymbol{\omega}) = \sum_{s=1}^{k}(\boldsymbol{\pi}'_t\mathbf{e}_s)R^q_{t:t+h}(\boldsymbol{\omega},s_t).$$

4. Calculate the $\alpha$ percentile, $VaR^{\alpha}_{t:t+h}$, of the resulting simulated distribution for multiperiod portfolio returns, $\{R^q_{t:t+h}(\boldsymbol{\omega})\}^Q_{q=1}$:

$$\frac{1}{Q}\sum_{q=1}^{Q} I_{\{R^q_{t:t+h}(\omega)\leq -VaR^{\alpha}_{t:t+h}\}} = \alpha,$$

where $I_{\{R^q_{t:t+h}(\omega)\leq -VaR^{\alpha}_{t:t+h}\}}$ is an indicator function that equals unity if $R^q_{t:t+h}(\omega) \leq -VaR^{\alpha}_{t:t+h}$.

Since some regimes occur relatively infrequently, we set the number of Monte Carlo simulations to a relatively large number, $Q = 50,000$.

Monte Carlo methods are also employed to calculate VaR when asset returns follow a GARCH process. For example, under a bivariate GARCH(1,1)-M process, we simply replace the first step of the algorithm by generating $Q$ $h-$period returns $\{\mathbf{r}^q_{t:t+h}\}^Q_{q=1}$ from a model of the form

$$\begin{aligned}
\mathbf{r}_{t+\tau,q} &= \boldsymbol{\xi} + \Lambda vech(\Omega_{t+\tau,q}) + \boldsymbol{\psi}_{t+\tau,q}\boldsymbol{\varepsilon}_{t+\tau,q} \\
\Omega_{t+\tau,q} &= \begin{bmatrix} b_{01} + b_{11}\eta^2_{1,t+\tau-1,q} + b_{12}\sigma^2_{11,t+\tau-1,q} & \cdot \\ \rho\sigma_{11,t+\tau-1,q}\sigma_{22,t+\tau-1,q} & b_{02} + b_{12}\eta^2_{1,t+\tau-1,q} + b_{22}\sigma^2_{22,t+\tau-1,q} \end{bmatrix}
\end{aligned}$$

where $\boldsymbol{\psi}'_{t+\tau,q}\boldsymbol{\psi}_{t+\tau,q} = \Omega_{t+\tau,q}$, $\mathbf{r}^q_{t:t+h} \equiv \sum_{\tau=1}^{h}\mathbf{r}_{t+\tau,q}$, $\boldsymbol{\varepsilon}_{t+\tau,q}$ is drawn from the appropriate IID distribution (e.g. a bivariate student-$t$). Simulations from univariate component GARCH models follows similar steps. Unknown parameter values are replaced by their estimates from Section 2.

### 3.3. *Empirical Results*

For each of the econometric methods, Figure 1 shows *unconditional* VaR term structures at the 1% level for horizons extending from a single month to two years. These plots assume that the initial state is drawn from the ergodic distribution for the model under consideration, i.e. using the ergodic state probabilities ($\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$) for the Markov switching model and average variance as initial values for the GARCH models. We use the convention of reporting $VaR$ as a positive number representing a loss. Several interesting findings emerge. First, there is very significant variation in the VaR estimates across the models included in this study, particularly at the longer horizons. For example, for the fifty-fifty stock-bond portfolio the 1% VaR ranges from 5% to 7% at the one month horizon and from 8% to 28% at the 24 month horizon.

At the longest horizons the bivariate GARCH(1,1)-M model with student-t errors leads to the highest estimates of VaR for most of the portfolios. The smallest VaR estimates are generally produced by the Gaussian IID model although lower estimates are implied by the Markov switching model fitted to the bond portfolios. The GARCH components models generate VaR term structures with steep slopes at short horizons that peak after 10 to 16 months and produce the highest VaR at short horizons for most portfolios. The non-parametric bootstrap VaR estimates generally lie in between the parametric models.

The ordering of the VaR estimates across different methods is far from invariant to the forecast horizon. For example, for the pure stock and mixed stock and T-bill portfolios the Markov switching model and GARCH(1,1)-M model produce relatively low VaR estimates at the shortest one month horizon, yet these models also produce the highest VaR estimates at the longest two-year horizon.

In practice interest lies in calculating *conditional* term structures given current information. To shed light on the conditional term structures implied by our parametric models, Figure 2 plots VaR term structures starting plus or minus one standard deviation away from the average volatility level (dotted lines) in the case of the GARCH models, or from each of the four states in the case of the Markov switching model. This figure shows that the changes in VaR levels and term structure shapes across the four regimes capture wider variation than that found by varying the initial volatility level for the GARCH models. VaR estimates for the stock portfolios are generally very high when starting from the low-mean, high-volatility state (state 1) and conversely very low and flat when starting from the third state with high mean returns. Conditional VaR estimates computed under the Markov switching model are relatively conservative for bond portfolios when starting from the regimes with low bond volatility (regimes two and three).

Notice that the initial state matters significantly for the VaR estimates even at the longest horizons considered here. When $h \to \infty$, $\mathbf{r}_{t+h}$ must come from the ergodic distribution characterized by the state probabilities $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$. However, the VaR depends on the cumulated return, $\sum_{\tau=1}^{h} \mathbf{r}_{t+\tau}$, and thus reflects the initial state. One should therefore expect the lines starting from the four states in Figure 2 to become parallel when $h \rightarrow \infty$, since most of the returns contributing to the cumulants come from very similar distributions at longer horizons.

### 3.4. *Expected shortfall*

Expected shortfall is another commonly reported measure of risk. It is defined as the expected loss conditional on the (cumulated) loss exceeding the VaR. At the $h$-period horizon, the conditional expected shortfall is hence given by

$$ES^{\alpha}_{t:t+h} = E[(R_{t:t+h}|R_{t:t+h} \leq -VaR^{\alpha}_{t:t+h})|\mathcal{F}_t].$$

When single-period portfolio returns are $IIN(\mu, \sigma^2)$, the $h-$period expected shortfall is given by

$$ES^{\alpha}_{t:t+h} = \mu h - \sigma\sqrt{h}\frac{\phi(\Phi^{-1}(\alpha))}{\alpha}. \tag{11}$$

Using again the convention of reporting a negative return as a positive loss, this peaks at $h^{**} = \sigma^2\phi(\Phi^{-1}(\alpha))^2/(4\mu^2\alpha^2)$ and declines linearly in $h$ at long horizons.

For the nonlinear econometric models we resort once more to simulations to calculate the expected shortfall. The algorithm calculates the conditional sample mean subject to the condition that portfolio returns fall in the $\alpha$ percentile to obtain an estimate of $E[R^q_{t:t+h}(\omega)|R^q_{t:t+h}(\omega) \leq -VaR^{\alpha}_{t:t+h}]$:

$$\frac{1}{\sum_{q=1}^{Q} I_{\{R^q_{t:t+h}(\boldsymbol{\omega}) \leq -VaR^{\alpha}_{t:t+h}\}}} \cdot \sum_{q=1}^{Q} I_{\{R^q_{t:t+h}(\boldsymbol{\omega}) \leq -VaR^{\alpha}_{t:t+h}\}} R^q_{t:t+h}(\boldsymbol{\omega}). \tag{12}$$

Estimates of the *unconditional* (or average) expected shortfall are provided in Figure 3. Differences across econometric models are even larger than those observed in the corresponding VaR figures (Figure 1). This is perhaps unsurprising given the different tail behavior assumed by these models. The GARCH models generate the highest shortfall estimates for the bond portfolios while the Markov switching and GARCH(1,1)-M model generate the largest estimates for the pure stock portfolio. The Gaussian IID model still produces relatively low estimates of the expected shortfall as does the nonparametric bootstrap method.

Figure 4 compares the *conditional* expected shortfall under the various econometric specifications, calculated under assumptions about the initial state identical to those in Figure 2. The picture emerging from this figure is again one of very large variations in term structures across models depending on the initial state and the form of the econometric model with particularly large variation observed across the four states in the Markov switching model, reflecting the very different properties of stock and bond returns across these states.[10]

The large differences in term structures of risk measures observed across different econometric specifications suggest that the choice of econometric model has important risk management implications. To assist in evaluating how good the various models are we next analyze their out-of-sample predictive performance.

## 4. **Econometric Tests of out-of-sample performance**

Econometric specifications used in risk management are best judged by their out-of-sample performance. This provides an appropriate method to control for overfitting, which could be a concern for some of the heavily parameterized approaches considered here. To assess the methods' out-of-sample forecasting performance we report the unconditional coverage computed as the percentage of periods where the $\alpha\%$ VaR is exceeded. We also report the p-values of the $S$-statistic suggested by Christoffersen and Diebold (2000) which tests for predictability of the 'hit' sequence of indicator variables tracking whether the VaR is exceeded. To enhance statistical power we compute this statistic using overlapping data with finite-sample critical values generated using the stationary bootstrap proposed by Politis and Romano (1994).[11] Ideally one would want the unconditional coverage to be as close to the VaR level as possible and the hit indicator function to be unpredictable.

### 4.1. *Empirical Results*

We recursively estimate all the parameters of the models introduced in Section 3 and proceed to calculate the VaR at all points in time between 1980:01 and 1999:12. For the Markov switching model this implies re-estimating all parameters and the state probability vector $\boldsymbol{\pi}_t$ on an expanding window of data using the EM algorithm. For other models, only the parameters are estimated recursively by MLE. The block bootstrap is simply re-applied at each point in time as new data becomes available. At each point in time simulation methods are then used to calculate conditional

[10] In a related study Christoffersen and Diebold (2003) analyse the importance of predictability in first and second moments of returns at different horizons and find that sign predictability is horizon-specific, tending to be strongest at intermediate horizons.

[11] In view of the limited length of the out-of-sample period we do not report results for the expected shortfall measure, but see Christoffersen (2003) for a discussion of possible tests.

risk measures.

Empirical results are reported in Table 1 for $\alpha = 1\%$ and $\alpha = 5\%$ using horizons of $h = 1$, 4, 6 and 12 months. The GARCH(1,1)-M model overestimates VaR for the bond portfolios and has zero coverage probabilities at all horizons for the pure bond portfolio. If the VaR measure is used to determine capital requirements, this model would lead to excessively high reserves. The GARCH component model is much better at short horizons but suffers from the opposite problem at longer horizons where it underestimates the risk of the bond portfolios. The IID Gaussian model tends to underestimate the tail risk especially at the longer horizons ($h \geq 6$ months) and for the stock portfolios. VaR estimates from the Markov switching model are generally quite good at $\alpha = 1\%$ but are too low when $\alpha = 5\%$, particularly at the longest horizons. The bootstrap produces nearly perfect coverage probabilities at the 5% level.

Using the $S$-statistic suggested by Christoffersen and Diebold (2000), there is strong evidence of predictability in the hit sequence generated by the component GARCH models. This occurs across horizons and portfolios. Evaluation of the GARCH(1,1)-M student-$t$ model is made difficult by the zero coverage probabilities produced by this model. At the 1% VaR level the Markov switching model does not produce any significant rejections of the null of no predictability of the hit indicator, while both the IID Gaussian and bootstrap methods do so in a number of cases. Conversely, when $\alpha = 5\%$, the bootstrap method produces the fewest violations of the null.

At the 1% VaR level the best results overall are found for the Markov switching model, while the nonparametric bootstrap method gives the best results for a VaR level of $\alpha = 5\%$. The good performance of the regime switching model for risks in the far tail ($\alpha = 1\%$) is likely to be related to the fact that it is a mixture of states with one state centered on a large negative mean return (state 1). Conversely, the block bootstrap method seems to work well a little further away from the extreme tail.

## 5. **Conclusion**

Our analysis considered implications for the term structure of risk of a variety of stock and bond portfolios under a range of parametric econometric specifications widely used in the literature in addition to a non-parametric bootstrap approach. The large differences in term structures of risk measures observed across different approaches suggest that the choice of econometric model has important risk management implications. Our out-of-sample forecasting results suggested that no model clearly dominates, the best performing model depending on the horizon, the portfolio considered, and the VaR level, $\alpha$. The regime switching model performed quite well for far tail risks ($\alpha = 1\%$) while the non-parametric bootstrap produced the best performance a bit further away from the far left tail at $\alpha = 5\%$.

Our analysis studied risk measures under the assumption of fixed portfolio weights. An important question is how the risk measures analyzed in this paper could be used for portfolio construction. As an illustration, consider an investor with preferences described by a utility function trading off expected returns against expected shortfall, $E[R_{t:t+h}] - ES^{\alpha}_{t:t+h}$. Using estimates from our econometric models, Figure 5 plots utility term structures for this function using steady state probabilities or average volatility figures as a starting point. Under the regime switching model a very conservative portfolio with 50% bonds and 50% T-bills is preferred by this investor at the one-month horizon ($h = 1$) while for $h = 24$ months a portfolio that invests 50% in bonds and 50% in stocks is preferred. Interestingly, these rankings appear to be model-specific. For example at the two-year horizon ($h = 24$) under the component GARCH model a 100% bond portfolio is selected, while under the GARCH(1,1)-M student-$t$ model the preferred portfolio has 50% in stocks and 50% in cash. Clearly the choice of econometric model has implications not only for the term structure of risk measures but also for asset allocation choices that trade off expected returns against estimated risk. We leave a more detailed analysis of this question to future analysis.

Many additional questions emerge for future research. We ignored parameter estimation uncertainty in our analysis, but this could have important effects on the results. An easy - but computationally intensive - way to address this is by drawing parameter values from their asymptotic distribution and repeatedly simulating risk measures conditional on these values to form confidence intervals. Another possibility is to extend the regime switching models analyzed here by considering mixtures of non-Gaussian distributions or by including leverage effects.

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## Table 1

## Out-of-Sample Tests of Predictive Accuracy

The table reports the results of out-of-sample tests concerning Value at Risk measures for multi-period portfolio returns. The predictive accuracy measures consist of the unconditional coverage probability and Christoffersen and Diebold's (2000) *S* test capturing deviations from iid-ness (persistence) of an indicator variable $I_t$ that records portfolio returns below the VaR. The p-value for the *S* statistic is obtained using bootstrap methods on overlapping portfolio returns. In the table, MS stands for multivariate Markov Switching, t-GARCH(p,q)-M for bivariate GARCH(p,q)-in mean with t-distributed errors, C-GARCH(p,q)-M for component GARCH(p,q)-in mean, and 'Bootstrap' for an historical VaR calculated using a stationary bootstrap algorithm with block length matching the forecast horizon *h*.

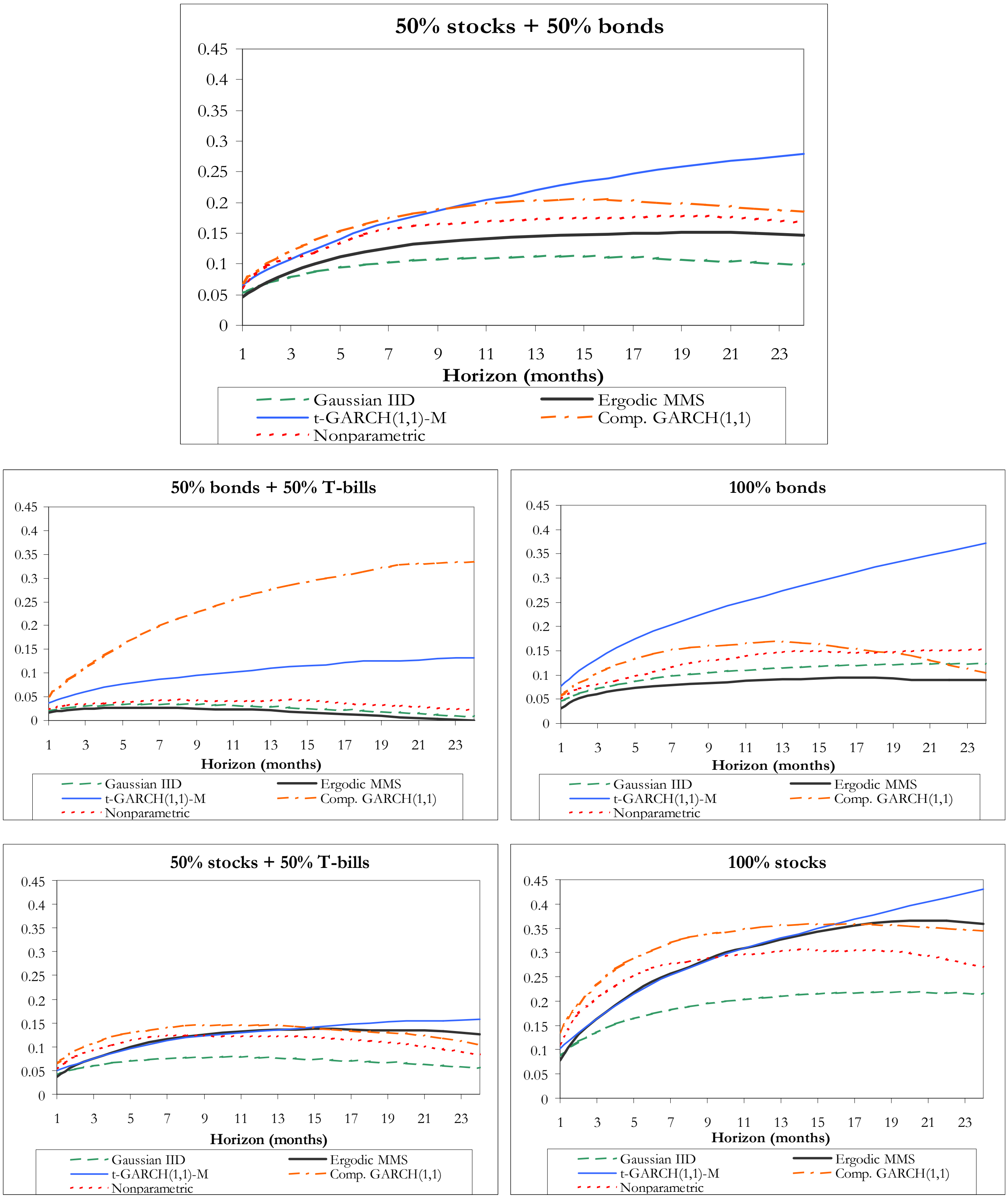
| **Panel I – 1% Value-at-Risk** | **50% Bonds+50% T-bills** | | | | **100% Bonds** | | | | **50% Stocks+50% T-bills** | | | | **50% Stocks+50% Bonds** | | | | **100% Stocks** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **h=1** | **h=4** | **h=6** | **h=12** | **h=1** | **h=4** | **h=6** | **h=12** | **h=1** | **h=4** | **h=6** | **h=12** | **h=1** | **h=4** | **h=6** | **h=12** | **h=1** | **h=4** | **h=6** | **h=12** |
| A. Unconditional coverage probability | | | | | | | | | | | | | | | | | | | | |
| MS | 0.017 | 0.013 | 0.021 | 0.048 | 0.017 | 0.013 | 0.021 | 0.048 | 0.013 | 0.013 | 0.004 | 0.000 | 0.008 | 0.017 | 0.004 | 0.013 | 0.013 | 0.017 | 0.004 | 0.000 |
| t-GARCH-M | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.008 | 0.008 | 0.013 | 0.009 | 0.004 | 0.000 | 0.000 | 0.004 | 0.008 | 0.008 | 0.013 | 0.009 |
| C-GARCH-M | 0.004 | 0.004 | 0.013 | 0.171 | 0.004 | 0.021 | 0.038 | 0.127 | 0.008 | 0.013 | 0.013 | 0.013 | 0.008 | 0.000 | 0.000 | 0.004 | 0.008 | 0.030 | 0.021 | 0.018 |
| Bootstrap | 0.013 | 0.004 | 0.009 | 0.031 | 0.013 | 0.004 | 0.009 | 0.031 | 0.013 | 0.008 | 0.009 | 0.009 | 0.004 | 0.008 | 0.000 | 0.013 | 0.008 | 0.008 | 0.004 | 0.009 |
| IID Gaussian | 0.013 | 0.013 | 0.009 | 0.039 | 0.013 | 0.013 | 0.009 | 0.039 | 0.008 | 0.025 | 0.030 | 0.026 | 0.008 | 0.025 | 0.030 | 0.044 | 0.008 | 0.025 | 0.030 | 0.026 |
| B. *S*-statistic (p-value) | | | | | | | | | | | | | | | | | | | | |
| MS | 0.329 | 0.136 | 0.999 | 0.985 | 0.329 | 0.136 | 0.999 | 0.985 | 0.345 | 0.336 | 0.411 | NA† | 0.838 | 0.114 | 0.071 | 0.751 | 0.345 | 0.336 | 0.411 | NA† |
| t-GARCH-M | NA† | NA† | NA† | NA† | NA† | NA† | NA† | NA† | 0.999 | 1.000 | 1.000 | NA† | 1.000 | NA† | NA† | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| C-GARCH-M | 0.472 | 0.070 | 0.029 | 0.000 | 0.484 | 0.001 | 0.000 | 0.007 | 0.797 | 0.086 | 0.048 | 0.063 | 0.670 | NA† | NA† | 0.308 | 0.852 | 0.083 | 0.176 | 0.000 |
| Bootstrap | 0.510 | 0.028 | 0.032 | 0.140 | 0.510 | 0.028 | 0.032 | 0.140 | 0.710 | 0.181 | 0.133 | 0.216 | 0.198 | 0.010 | NA† | 0.221 | 0.854 | 0.181 | 0.268 | 0.261 |
| IID Gaussian | 0.478 | 0.201 | 0.040 | 0.249 | 0.478 | 0.201 | 0.040 | 0.249 | 0.793 | 0.072 | 0.090 | 0.137 | 0.656 | 0.064 | 0.093 | 0.190 | 0.793 | 0.072 | 0.090 | 0.137 |
| **Panel II – 5% Value-at-Risk** | | | | | | | | | | | | | | | | | | | | |
| A. Unconditional coverage probability | | | | | | | | | | | | | | | | | | | | |
| MS | 0.059 | 0.097 | 0.132 | 0.232 | 0.067 | 0.093 | 0.141 | 0.232 | 0.075 | 0.064 | 0.068 | 0.066 | 0.071 | 0.072 | 0.085 | 0.127 | 0.075 | 0.064 | 0.068 | 0.066 |
| t-GARCH-M | 0.013 | 0.000 | 0.000 | 0.000 | 0.013 | 0.000 | 0.000 | 0.000 | 0.092 | 0.085 | 0.081 | 0.053 | 0.046 | 0.047 | 0.047 | 0.039 | 0.092 | 0.085 | 0.081 | 0.053 |
| C-GARCH-M | 0.013 | 0.047 | 0.162 | 0.461 | 0.017 | 0.064 | 0.128 | 0.364 | 0.025 | 0.055 | 0.064 | 0.083 | 0.017 | 0.021 | 0.030 | 0.039 | 0.038 | 0.068 | 0.081 | 0.123 |
| Bootstrap | 0.029 | 0.047 | 0.056 | 0.066 | 0.029 | 0.047 | 0.056 | 0.066 | 0.046 | 0.051 | 0.056 | 0.044 | 0.033 | 0.047 | 0.051 | 0.053 | 0.046 | 0.051 | 0.056 | 0.044 |
| IID Gaussian | 0.033 | 0.064 | 0.107 | 0.215 | 0.033 | 0.064 | 0.107 | 0.215 | 0.054 | 0.085 | 0.094 | 0.132 | 0.046 | 0.097 | 0.115 | 0.167 | 0.054 | 0.085 | 0.094 | 0.132 |
| B. *S*-statistic (p-value) | | | | | | | | | | | | | | | | | | | | |
| MS | 0.082 | 0.001 | 0.137 | 0.132 | 0.157 | 0.000 | 0.007 | 0.242 | 0.031 | 0.061 | 0.159 | 0.051 | 0.009 | 0.384 | 0.309 | 0.024 | 0.031 | 0.061 | 0.159 | 0.051 |
| t-GARCH-M | 0.999 | NA† | NA† | NA† | 0.999 | NA† | NA† | NA† | 0.010 | 0.232 | 0.257 | 0.117 | 0.004 | 0.107 | 0.074 | 0.142 | 0.010 | 0.232 | 0.257 | 0.117 |
| C-GARCH-M | 0.469 | 0.097 | 0.240 | 0.145 | 0.453 | 0.107 | 0.001 | 0.282 | 0.565 | 0.090 | 0.148 | 0.208 | 0.462 | 0.003 | 0.005 | 0.025 | 0.347 | 0.419 | 0.020 | 0.267 |
| Bootstrap | 0.415 | 0.425 | 0.258 | 0.120 | 0.415 | 0.425 | 0.258 | 0.120 | 0.002 | 0.337 | 0.246 | 0.200 | 0.010 | 0.107 | 0.058 | 0.146 | 0.002 | 0.337 | 0.246 | 0.200 |
| IID Gaussian | 0.328 | 0.000 | 0.001 | 0.245 | 0.328 | 0.000 | 0.001 | 0.245 | 0.001 | 0.350 | 0.089 | 0.207 | 0.000 | 0.389 | 0.375 | 0.082 | 0.001 | 0.350 | 0.089 | 0.207 |

† = No multi-period portfolio returns are below the VaR measure in the sample.

## Figure 1

## 1% (Unconditional) VaR

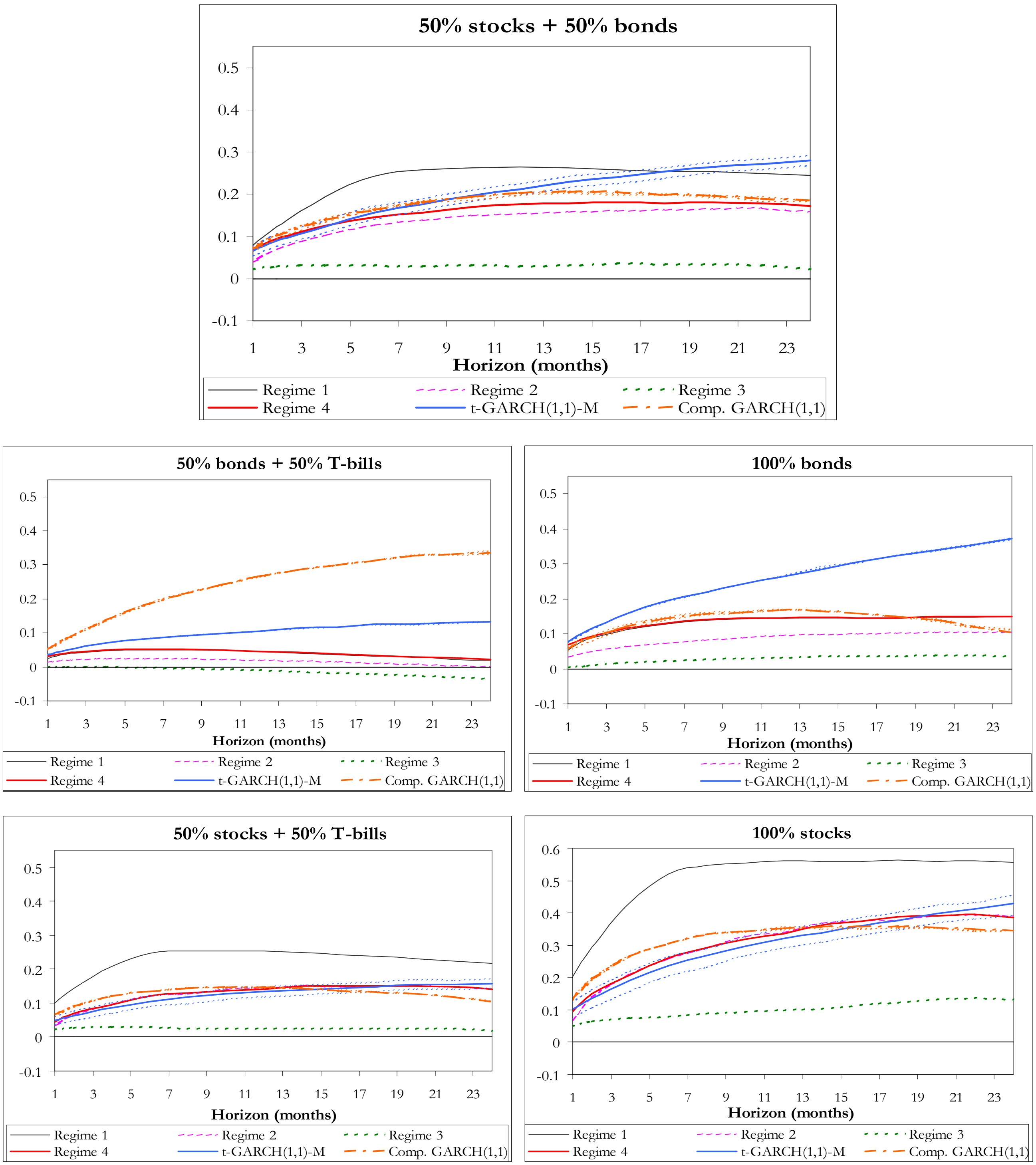
The graphs plot the (negative of the) first percentile of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon. The regime-switching VaR is calculated under the assumption the current vector of state probabilities correspond to the vector of ergodic (long-run) probabilities.



## Figure 2

## 1% Conditional VaR – Effects of the Horizon and of the Initial State

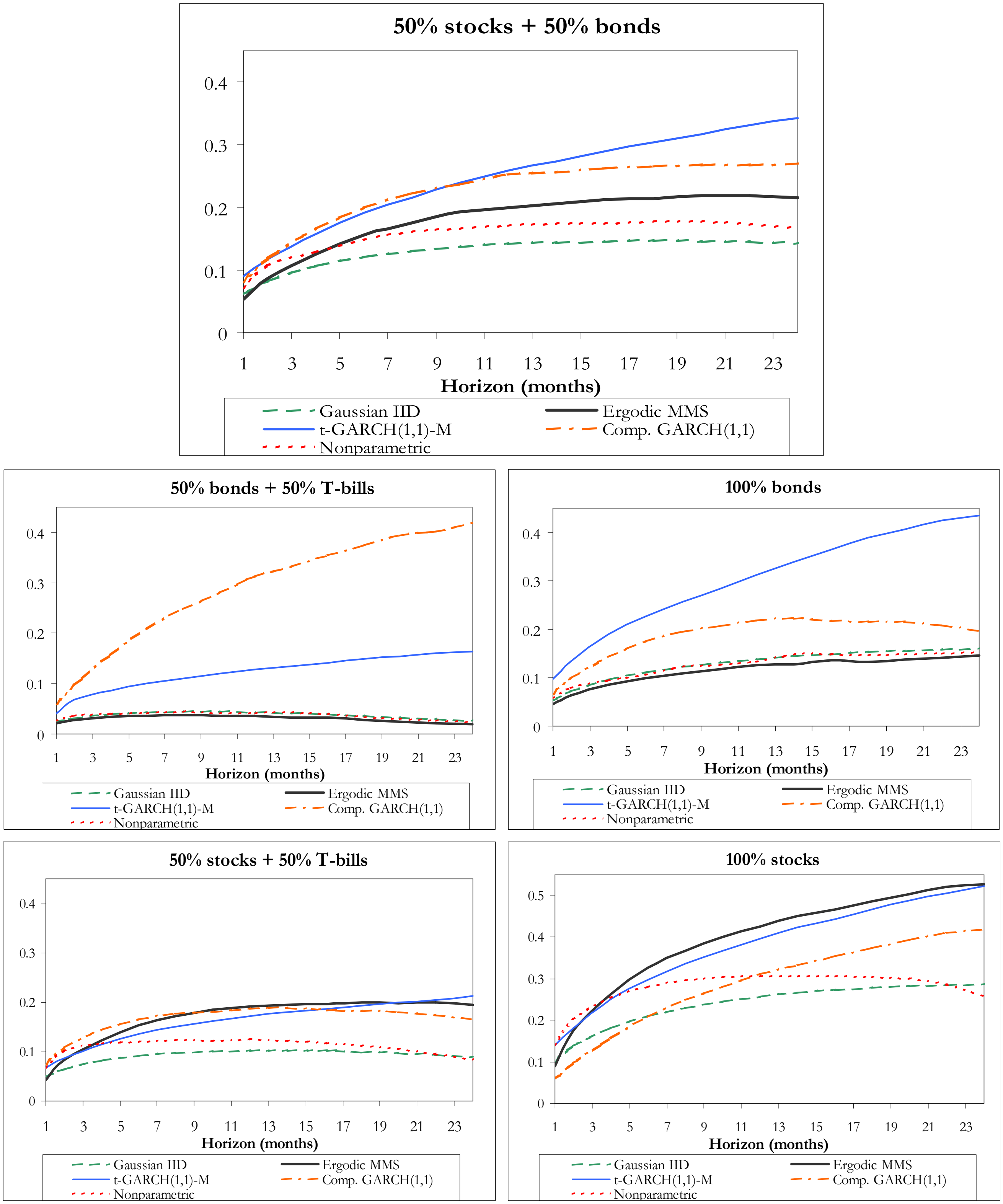
The graphs plot the (negative of the) first percentile of the distribution of returns for various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon and of the current state. For the GARCH-type models, term structure schedules are plotted for three cases: (i) initial variance(s) are set equal to the long-run (steady-state) value; (ii) initial variance(s) are set equal to the average variance minus one standard deviation; (iii) initial variance(s) are set equal to the average variance plus one standard deviation. Term structure schedules corresponding to cases (ii) and (iii) are represented as dotted curves.



## Figure 3

# 1% Unconditional Expected Shortfall

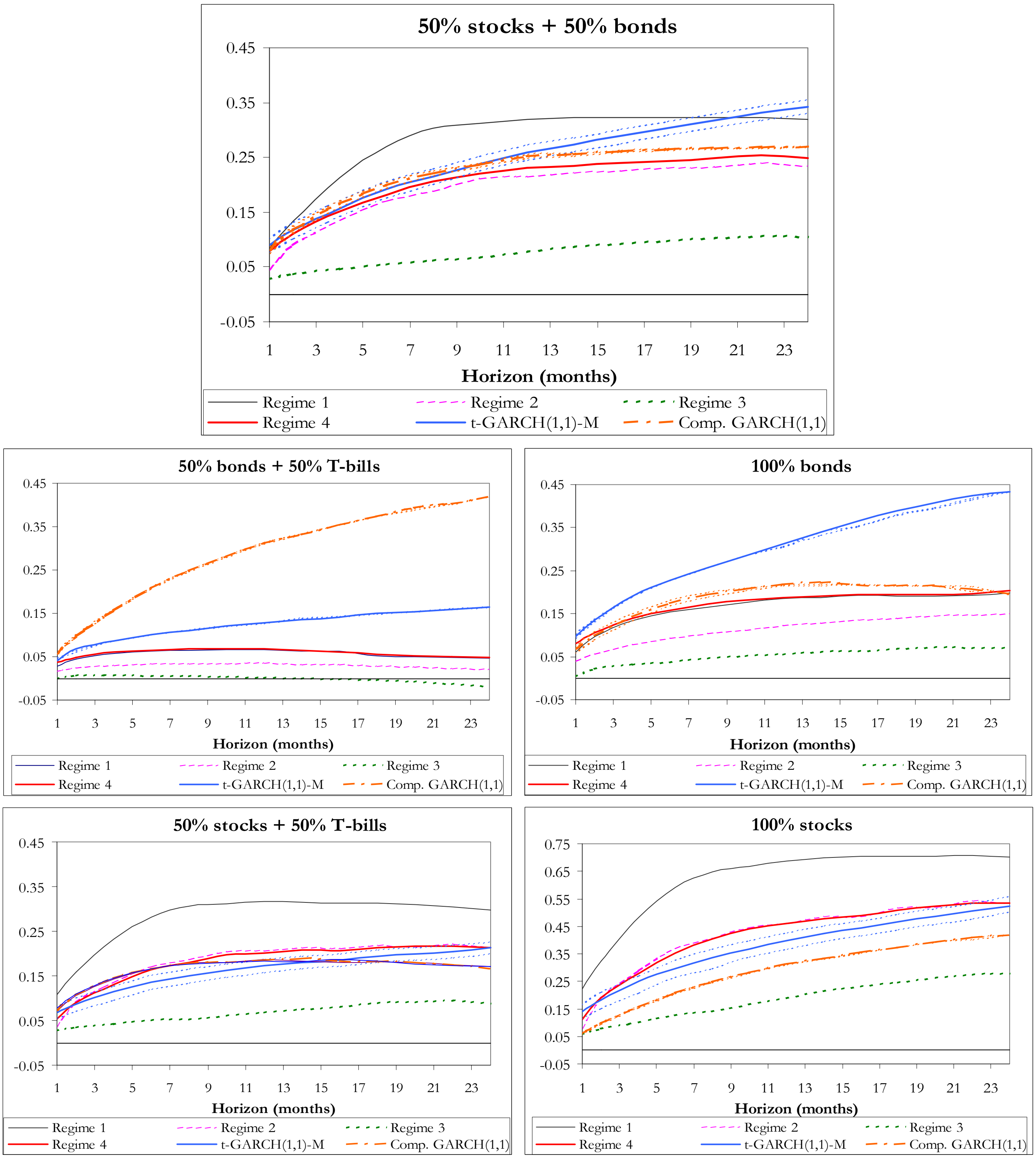
The graphs plot the 1% shortfall of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon. The regime-switching VaR is calculated under the assumption the current vector of state probabilities correspond to the vector of ergodic (long-run) probabilities.



## Figure 4

## 1% Conditional Expected Shortfall – Effects of the Horizon and of Initial State

The graphs plot the 1% shortfall for various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon and of the current state. For the GARCH-type models, term structure schedules are plotted for three cases: (i) initial variance(s) are set equal to the long-run (steady-state) value; (ii) initial variance(s) are set equal to the average variance minus one standard deviation; (iii) initial variance(s) are set equal to the average variance plus one standard deviation. Term structure schedules corresponding to cases (ii) and (iii) are represented as dotted curves.



## Figure 5

## Unconditional Risk-Adjusted Expected Return

The graphs plot the mean portfolio return in excess of the 5% expected shortfall of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon. The regime-switching values are calculated under ergodic probabilities.