Strategic Asset Allocation and Consumption Decisions under Multivariate Regime Switching*

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Abstract

This paper studies strategic asset allocation and consumption choice in the presence of regime switching in asset returns. We find evidence that four separate regimes - characterized as crash, slow growth, bull and recovery states - are required to capture the joint distribution of stock and bond returns. Optimal asset allocations vary considerably across these states - both among bonds and stocks and among large and small stocks - and change over time as investors revise their estimates of the underlying state probabilities. In the crash state investors always allocate more of their portfolio to stocks the longer their investment horizon, while the optimal allocation to stocks declines as a function of the investment horizon in bull markets. The joint effects of learning about the underlying state probabilities and predictability of asset returns from the dividend yield give rise to a non-monotonic relationship between the investment horizon and the demand for stocks. Consumption-to-wealth ratios are found to depend on the underlying state and welfare costs from ignoring regime switching are substantial even after accounting for parameter uncertainty. Out-of-sample forecasting experiments confirm the economic importance of accounting for the presence of regimes in asset returns.

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1. **Introduction**

For most investors the strategic asset allocation decision−how much to invest in major asset classes such as cash, stocks and bonds−is a key determinant of their portfolio performance. The importance of this decision has further been highlighted by empirical findings suggesting that stock and bond returns contain a sizeable predictable component that introduces time-variations in investment opportunities and gives rise to a large hedging demand for multiperiod investors.[1]

Strategic asset allocation decisions can only be made in the context of a model for the joint distribution of asset returns. Most studies assume that asset returns are generated by a linear process with stable coefficients so the predictive power of state variables such as dividend yields, default and term spreads does not vary over time. However, there is mounting empirical evidence that asset returns follow a more complicated process with multiple "regimes", each of which is associated with a very different distribution of asset returns. Ang and Bekaert (2001, 2002), Ang and Chen (2002), Connolly, Stevers, and Sun (2004), Garcia and Perron (1996), Gray (1996), Guidolin and Timmermann (2004a,b, 2005a), Perez-Quiros and Timmermann (2000), Turner, Startz and Nelson (1989) and Whitelaw (2001) all report evidence of regimes in stock or bond returns.

In this paper we characterize investors' strategic asset allocation and consumption decisions under a regime-switching model for asset returns with four states characterized as crash, slow growth, bull and recovery states. A difference to earlier studies is that we allow the underlying states to be unobservable to the investor who must infer the state probabilities from the sequence of returns data. Regime switching means that all conditional moments of the asset return distribution are time-varying, so we extend the previous literature on strategic asset allocation to cover the case where all moments may need appropriate hedging. We find evidence that four separate regimes−characterized as crash, slow growth, bull and recovery states−are required to capture the joint distribution of stock and bond returns. However, none of the states can be perfectly anticipated: starting from any one of the states the investor always assigns a positive and non-negligible probability to the possibility of transitioning to a different state. This is particularly important for the crash state which is transitory−its average duration is only two months. Therefore its presence in the data is important for asset allocation purposes without being inconsistent with equilibrium arguments restricting the equity premium to be positive.

We show that the optimal asset allocation differs strongly across regimes. For instance, stocks are attractive to short-to-medium term investors in the bull state since the probability of staying in such a state is high. Stocks are far less attractive in the crash state even though this state is not very persistent. Even if, as seems plausible, investors never know with certainty which regime the economy is currently in, beliefs about state probabilities become important to the asset allocation.

Our paper is part of a growing literature that explores the asset allocation and utility cost implications of return predictability from the perspective of a small, expected utility maximizing investor with a multi-period horizon. In an analysis involving a single risky stock portfolio, Kandel and Stambaugh (1996) find that predictability can be statistically small yet still have a large effect on the optimal asset allocation.

[1] See, e.g., Brandt, Goyal, and Santa-Clara (2002), Brennan and Xia (2002), Campbell and Viceira (1999, 2001), Chacko and Viceira (2000), Cocco et al. (2001), Gerard and Wu (2002), Kandel and Stambaugh (1996) and Xia (2001). Campbell and Viceira (2002) is a milestone in this area and provides a comprehensive treatment of strategic asset allocation.

Barberis (2000) extends this result to long horizons. Campbell and Viceira (1999) derive closed-form expressions using log-linear approximations for a consumption and portfolio choice problem with continuous rebalancing and infinite horizon. Balduzzi and Lynch (1999) find that return predictability continues to affect optimal asset allocations and utility costs in the presence of realistic transaction costs. Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2001) and Campbell, Chan and Viceira (2003) study strategic asset allocation and document large effects of predictability on asset holdings and welfare costs.

One of the key questions addressed in the literature on optimal asset allocation is how the investment horizon affects optimal portfolio weights. When investment opportunities remain constant over time, a power utility investor's horizon does not affect the optimal asset allocation, c.f. Samuelson (1969). In the absence of predictor variables, standard models therefore imply constant portfolio weights. In contrast, using the dividend yield as a predictor, Barberis (2000) finds that the weight on stocks should increase as a function of the investor's horizon

Even in the absence of predictor variables, regime switching models imply that investors' asset allocation varies over time as the underlying states offer different investment opportunities and investors revise their beliefs about the state probabilities.[2] Horizon effects also vary across states. Since stocks are not very attractive in the crash state, investors with a short horizon hold very little in stocks in this state. At longer investment horizons, there is a high chance that the economy will switch to a better state and so investors allocate more towards stocks. In the crash state the allocation to stocks is therefore an increasing function of the investment horizon. In the more persistent slow growth and bull states, investors with a short horizon hold large positions in stocks. At longer horizons investment opportunities will almost certainly worsen so investors hold less in stocks, thereby creating a downward sloping relation between stock holdings and the investment horizon.

In addition to these horizon effects we find interesting substitution effects among small and large stocks. As the horizon expands, the allocation to small stocks as a proportion of the total equity portfolio typically declines, while the allocation to large stocks increases. This extends earlier findings that predictability of returns on small and large stocks can lead to important shifts in the composition of equity portfolios. Perez-Quiros and Timmermann (2000) use a bivariate model to capture regimes in the distribution of small and large stocks' returns and find that a simple stylized trading rule generates superior Sharpe ratios during recessions although they do not consider optimal asset allocation implications of regimes. Ang and Chen (2002) find that equity correlations that differ across high/low return states can be successfully captured by a regime switching model. They note that small firms' returns exhibit relatively strong asymmetries and argue that such asymmetric correlations may be important for strategic asset allocation purposes, although they stop short of analyzing this question.

Regime switching affects not only the optimal asset allocation but also the joint consumption and savings decision. For instance, a perception of being in a bull market induces investors to change current consumption since it changes both their perceived income and investment opportunities. In the crash state with poor investment opportunities, optimal consumption is relatively insensitive to the time horizon and uniformly below its steady-state value. Conversely, in the bull state investment opportunities are very good and income effects lead to a higher consumption-wealth ratio.

We extend the regime switching model for asset returns to include predictability from state variables

[2] See Veronesi (1999) for a discussion of similar effects in a two-state asset pricing model.

such as the dividend yield. Compared to a benchmark with constant expected returns, predictability from the dividend yield in a linear vector autoregression (VAR) reduces risk at longer horizon and leads to an increased demand for stocks, the longer the investment horizon. In contrast, regime switching leads to a positive correlation between return innovations and shocks to future expected returns, thereby increasing risk and lowering the long-term demand for stocks compared to the benchmark model with no predictability. In the model that combines regime switching with predictability from the yield we see non-monotonic relationships between the allocation to stocks and the investment horizon: At short horizons the effect of regimes tends to dominate while at longer horizons the mean reverting component in returns tracked by the yield dominates and leads to an increasing demand for stocks.

Finally, to evaluate the economic significance of our results we examine the 'real time' out-of-sample performance of asset allocation rules based on both standard VARs that use the dividend yield as a predictor variable and regime switching models for the joint dynamics of stock and bond returns. Consistent with earlier findings in the literature (e.g., Campbell, Chan and Viceira (2003)), we find that the recursively updated portfolio weights vary significantly over time as a result of changing investment opportunities and that optimal asset holdings are sensitive to how predictability is modeled. When regimes are taken into account, there is evidence that the allocation to stocks and bonds as well as the division of stock holdings among small and large firms is quite different from that obtained under linear models of predictability in asset returns. Furthermore, we generally find that the average realized utility is highest for models that account for regime switching.

The two papers whose modeling approach is most closely related to ours are Ang and Bekaert (2002) and Detemple, Garcia and Rindisbacher (2003). In an important contribution to the literature, Ang and Bekaert (2002) use a two-state model to evaluate the claim that the home bias observed in holdings of international assets can be explained by return correlations that increase in bear markets. Assuming observable states, they find that optimal portfolio weights depend both on the current regime and on the investment horizon and that the cost of ignoring regime switching is of the same order of magnitude as the cost of ignoring foreign equities in the optimal portfolio. While our paper shares a similar regime switching setup, we address a very different question, namely a US investor's strategic asset allocation between bonds, stocks and cash. We find that a four-state model is required to capture the rich dynamics of the joint distribution of stock and bond returns. Furthermore, we model regimes as unobservable, calculate asset allocations under optimal filtering and therefore explicitly address the effects on hedging demands arising from investors' recursive updating in their beliefs about the underlying state probabilities. In our model investors therefore have to account for revisions in future beliefs when determining their current asset allocation. In this sense our paper extends the rational learning exercise in Barberis (2000) to cover multivariate regime switching.

Detemple, Garcia and Rindisbacher (2003) approach a wide class of portfolio choice problems in continuous time, including strategic asset allocation. Building on the widespread evidence that both interest rates and the market price of risk(s) follow non-linear processes, they investigate the asset allocation implications of non-linear predictability using simulation methods. They show that findings in the standard VAR framework−e.g., that the equity allocation should be higher the longer the investment horizon−may be overturned in the presence of non-linearities. They also find that adding the dividend yield as a predictor to their non-linear model changes the optimal portfolio weights very little. For reasons similar to these authors, we resort to Monte Carlo methods to solve for the optimal asset allocation. However, we explore the strategic asset allocation under a class of non-linear processes that is not nested in their framework, multivariate regime switching in stock and bond returns.

The plan of the paper is as follows. Section 2 introduces the multi-state model used to capture predictability and regime switching for asset returns and reports empirical findings. Section 3 sets up the investor's asset allocation problem while Section 4 presents empirical asset allocation results. Section 5 extends the model to allow for predictability from the dividend yield and Section 6 studies a joint consumption and asset allocation problem. Section 7 presents utility cost calculations, investigates the effect of parameter uncertainty and examines the out-of-sample performance of alternative asset allocation schemes based on different models for the joint distribution of asset returns. Section 8 concludes. Technical details are provided in appendices at the end of the paper.

2. **Asset Returns under Regime Switching**

A number of stylized features of asset returns have emerged from the empirical finance literature. Stock and bond returns are−to a limited extent−predictable (e.g., Campbell (1987), Fama and French (1988, 1989) and Keim and Stambaugh (1986)), their volatility clusters over time (e.g., Bollerslev, Chou, and Kroner (1992) and Glosten, Jagannathan, and Runkle (1993)) and correlations are not the same in bull and bear markets (e.g., Ang and Chen (2002) and Perez-Quiros and Timmermann (2000)). At shorter horizons, stock returns are also far from normally distributed and affected by occasional outliers. Campbell and Ammer (1993) and Fama and French (1989) have shown that variables found to forecast stock returns also predict bond returns.

Regime switching models can capture such properties of the return distribution. These models typically identify bull and bear regimes with very different mean, variance and correlations across assets, c.f. Maheu and McCurdy (2000). As the underlying state probabilities change over time this leads to time-varying expected returns, volatility persistence and changing correlations and predictability in higher order moments such as the skew and kurtosis. This is consistent with Aït-Sahalia and Brandt (2001) who argue that higher order moments of stock and bond returns are time-varying although different moments are typically predicted by different combinations of economic variables. The degree of predictability of mean returns can also vary significantly over time in regime switching models−a feature that seems present in stock return data, c.f. Bossaerts and Hillion (1999). Finally, regime switching models are capable of capturing even complicated forms of heteroskedasticity, fat tails and skews in the underlying distribution of returns, c.f. Timmermann (2000).[3]

To capture the possibility of regimes in the joint distribution of asset returns and predictor variables, consider an $(n+m)\times 1$ vector of asset returns in excess of the T-bill rate, $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{nt})'$ extended by a set of m predictor variables, $\mathbf{z}_t = (z_{1t}, ..., z_{mt})'$. Suppose that the mean, covariance and serial correlations in returns are driven by a common state variable, S_t, that takes integer values between 1 and k:

$$\begin{pmatrix} \mathbf{r}_t \\ \mathbf{z}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{s_t} \\ \boldsymbol{\mu}_{zs_t} \end{pmatrix} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \begin{pmatrix} \mathbf{r}_{t-j} \\ \mathbf{z}_{t-j} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{zt} \end{pmatrix}. \tag{1}$$

[3] Another attractive property of regime switching models comes from their interpretation as mixtures of normals. These have been widely used to approximate densities of arbitrary form, c.f. Marron and Wand (1992).

Here $\boldsymbol{\mu}_{s_t}$ and $\boldsymbol{\mu}_{zs_t}$ are intercept vectors for $\mathbf{r}_t$ and $\mathbf{z}_t$ in state s_t, $\{\mathbf{A}_{j,s_t}\}_{j=1}^{p}$ are $(n+m)\times(n+m)$ matrices of autoregressive coefficients in state s_t, and $(\boldsymbol{\varepsilon}_t' \ \boldsymbol{\varepsilon}_{zt}')' \sim N(0, \boldsymbol{\Omega}_{s_t})$, where $\boldsymbol{\Omega}_{s_t}$ is an $(n+m\times n+m)$ covariance matrix. When $k=1$, equation (1) simplifies to a standard vector autoregression. Our model thus nests as a special case the standard linear (single-state) model used in much of the asset allocation literature. This model gets selected if the data only supports a single regime.

Regime switches in the state variable, S_t, are assumed to be governed by the transition probability matrix, $\mathbf{P}$, with elements

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i, j = 1, .., k. \tag{2}$$

Each regime is thus the realization of a first-order Markov chain with constant transition probabilities. Importantly, S_t is not observable and state probabilities must be inferred (e.g. using the Hamilton-Kim filter) from time series data on r_t and z_t.

While simple, this model is quite general and allows means, variances and correlations of asset returns to vary across states. Hence the risk-return trade-off can vary across states in a way that may have strong asset allocation implications. For example, knowing that the current state is a persistent bull state will make most risky assets more attractive than in a bear state.

Estimation proceeds by optimizing the likelihood function associated with (1)-(2). Since the underlying state variable, S_t, is unobserved we treat it as a latent variable and use the EM algorithm to update our parameter estimates, c.f. Hamilton (1989).

2.1. *Data*

Our analysis considers a US investor's asset allocation among three major asset classes, namely stocks, bonds and T-bills. We further divide the stock portfolio into large and small stocks in light of the empirical evidence suggesting that these stocks have very different risk and return characteristics that vary across different regimes, c.f. Ang and Chen (2002) and Perez-Quiros and Timmermann (2000).

Our analysis uses monthly returns on all common stocks listed on the NYSE, AMEX and NASDAQ. The first and second size-sorted CRSP decile portfolios are used to form a portfolio of small firm stocks, while deciles 9 and 10 are used to form a portfolio of large firm stocks. We also consider the return on the CRSP portfolio of 10-year T-bonds. Returns are continuously compounded and inclusive of any cash distributions. To obtain excess returns we subtract the 30-day T-bill rate from these returns. Dividend yields are also used in the analysis and are computed as dividends on a value-weighted portfolio of stocks over the previous twelve month period divided by the current stock price. Our sample is January 1954 - December 1999, a total of 552 observations. Consistent with the literature (e.g. Barberis (2000) and Campbell, Chan, and Viceira (2003)) we only use data after the 1951 Treasury Accord. Data from 2000-2003 is not used for model selection or parameter estimation in order to keep a genuine post-sample period. All data is obtained from the Center for Research in Security Prices.

2.2. *Choice of Model Specification*

Guidolin and Timmermann (2005a) provide a specification analysis to determine the statistical evidence in support of regimes in the univariate and joint distribution of stock and bond returns. Considering a range of values for the number of states, $k = 1, 2, 3, 4, 5, 6$ and the lag-order $p = 0, 1, 2, 3$, they use information criteria to select a four state model. Single-state models or models with a smaller number of states get strongly rejected using a test such as that proposed by Garcia (1998). While there is evidence of two or three states in the separate distributions of stock and bond returns, the state variables are weakly correlated so a larger model with four states is required to capture the joint dynamics in stock and bond returns.

Our objective here is quite different since we are less concerned with statistical evidence and more interested in ensuring that the return distribution is correctly specified. To determine the optimal asset allocation, an investor has to compute expected utility which requires integrating over the return distribution implied by a particular model. If this model is misspecified, suboptimal asset allocation decisions will almost certainly follow, so it is important to make sure that the model is not misspecified.

We therefore use the predictive density specification tests proposed by Diebold et al. (1998) and Berkowitz (2001). These tests are based on the probability integral transform or $z-$score. This is the probability of observing a value smaller than or equal to the realization of returns, $\tilde{r}_{t+1}$, under the null that the model is correctly specified. Under the k-state mixture of normals, this is given by

$$\begin{aligned} \Pr(r_{t+1} &\leq \tilde{r}_{t+1}|\Im_t) = \sum_{i=1}^{k} \Pr(r_{t+1} \leq \tilde{r}_{t+1}|s_{t+1}=i,\Im_t)\Pr(s_{t+1}=i|\Im_t) \\ &= \sum_{i=1}^{k} \Phi\left(\sigma_i^{-1}(\tilde{r}_{t+1} - \mu_i - \sum_{j=1}^{p} a_{j,i} r_{t+1-j})\right)\Pr(s_{t+1}=i|\Im_t) \\ &\equiv z_{t+1}. \end{aligned} \quad (3)$$

Here r_t is the excess asset return, $\Im_t = \{\mathbf{r}_\tau \ \mathbf{z}_\tau\}_{\tau=1}^{t}$ is the information set at time t and $\Phi(\cdot)$ is the cumulative density function of a standard normal variable. If the model is correctly specified, z_{t+1} should be independently and identically distributed (IID) and uniform on the interval $[0,1]$, c.f. Rosenblatt (1952). Berkowitz (2001) proposes a likelihood-ratio test that inverts Φ to get a transformed $z-$score, $z_{t+1}^* = \Phi^{-1}(z_{t+1})$. Provided that the model is correctly specified, z^* should be IID and normally distributed ($IIN(0,1)$). We use a likelihood ratio test that focuses on a few salient moments of the return distribution. Suppose the log-likelihood function is evaluated under the null that $z_{t+1}^* \sim IIN(0,1)$:

$$L_{IIN(0,1)} \equiv -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T}\frac{(z_t^*)^2}{2}, \quad (4)$$

where T is the sample size. Under the alternative of a misspecified model, the log-likelihood function incorporates deviations from the null, $z_{t+1}^* \sim IIN(0,1)$:

$$z_{t+1}^* = \mu + \sum_{j=1}^{p}\sum_{i=1}^{l} \rho_{ji}(z_{t+1-i}^*)^j + \sigma e_{t+1}, \quad (5)$$

where $e_{t+1} \sim IIN(0,1)$. The null of a correct return model implies $p \times l + 2$ restrictions — i.e., $\mu = \rho_{ji} = 0$ ($j = 1,...,p$ and $i = 1,...,l$) and $\sigma = 1$ — in equation (5). Let $L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma})$ be the maximized log-likelihood obtained from (5). To test that the forecasting model (1)-(2) is correctly specified, we use the following test statistic

$$LR = -2\left[L_{IIN(0,1)} - L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma})\right] \sim \chi^2_{p\times l+2}. \quad (6)$$

In addition to the standard Jarque-Bera test that considers skew and kurtosis in the z-scores, we present three likelihood ratio tests, namely a test of zero-mean and unit variance ($p = l = 0$), a test of lack of serial correlation in the $z-$scores and a test that further restricts their squared values to be serially uncorrelated in order to test for omitted volatility dynamics.

Panel A of Table 1 shows the results of these tests for the three asset classes under consideration using a range of model specifications with up to six states. To detect the source of potential misspecifications the tests are applied separately to each asset class. Although none of the models passes all tests, the most parsimonious model that captures the distribution of both large stock returns and bond returns is a four-state model with regime-dependent mean and covariance matrix. Some aspects of small firms' return distribution are not captured by this model, but values of most of the test statistics tend to be quite small (albeit statistically significant). Models with fewer states or constant volatility across states produce very large values of the Jarque-Bera test, and are hence clearly mis-specified, while models with more states have far more parameters so we select a specification with four states. Interestingly, no VAR terms are required. This is consistent with the common finding that asset returns are only weakly serially correlated.

2.3. *Model Estimates*

Since both statistical and economic criteria for model specification suggest a four state model with regime-dependent means and variances, Figure 1 plots the state probabilities while Table 2 shows the parameter estimates for this model. Initially, we focus on the simplest case where $m = 0$ so no predictor variable is included to model the dynamics of asset returns.[4]

It is easy to interpret the four regimes. Regime 1 is a 'crash' state characterized by large, negative mean excess returns and high volatility. It includes the two oil price shocks in the 1970s, the October 1987 crash, the early 1990s, and the 'Asian flu'. Regime 2 is a low growth regime characterized by low volatility and small positive mean excess returns on all assets. Regime 3 is a sustained bull state where stock prices — especially those of the small firms— grow rapidly on average. Mean excess returns on long-term bonds are negative in this state. States 2 and 3 identify a size effect in stock returns. In state 2 the mean return of large stocks exceeds that of small stocks by about 7% per annum, while this gets reversed in state 3. Regime 4 is a recovery state with strong market rallies and high volatility for small stocks and bonds.

The negative expected return in regime 1 may seem extreme and appear to be incompatible with equilibrium arguments by which risky assets should earn a positive risk premium. This is not the case, however, due to the transitory nature of the crash state. The probability of leaving this regime for a state with positive expected returns exceeds 50% and on average the economy only spends two months in state 1. To see this, notice that, starting from the crash state, the conditional risk premium is given by

$$E[r_{it+1}|s_t = 1] = \sum_{j=1}^{k} E[r_{it+1}|s_{t+1} = j] \Pr(s_{t+1} = j|s_t = 1).$$

Using the estimated values of $\Pr(s_{t+1} = j|s_t = 1)$ and $E[r_{it+1}|s_{t+1} = j] \equiv \mu_{ij}$ reported in Table 2, we obtain conditional risk premia of 0.32, 0.01, and 0.003 percent for small stocks, large stocks, and bonds,

[4] Attempts to simplify the number of parameters by imposing the restriction that mean returns are the same across the four states or that the covariance matrices are identical in the high volatility states (states 1 and 4) were clearly rejected at critical levels below 1 percent, c.f. Guidolin and Timmermann (2003).

respectively.

Correlations between returns also appear to vary substantially across regimes. The estimated correlation between large and small firms' returns varies from a high of 0.82 in the crash state to a low of 0.50 in the recovery state. The correlation between returns on large stocks and bonds even changes signs across different regimes and varies from 0.37 in the recovery state to -0.40 in the crash state. Finally, the correlation between small stock and bond returns goes from -0.26 in the crash state to 0.12 in the slow growth state. This is consistent with the evidence of time-varying (regime-specific) correlations found in monthly equity portfolio returns by Ang and Chen (2002). The ability of our model to identify the negative correlation between stock and bond returns in the crash state−which the linear model is unable to do−is a sign of the potential value of adopting a multi-state model.[5]

Mean returns and volatilities are larger in absolute terms in the crash and recovery regimes, so it is perhaps unsurprising that the persistence of the states also varies considerably. The crash state has low persistence and on average only two months are spent in this regime. Interestingly, the transition probability matrix has a very particular form. Exits from the crash state are almost always to the recovery state and occur with close to 50 percent chance suggesting that, during volatile markets, months with large, negative mean returns cluster with months that have high positive returns. The slow growth state is far more persistent with an average duration of seven months while the bull state is the most persistent state with an average duration of eight months. Finally, the recovery state is again not very persistent and the market is expected to stay just over three months in this state. The steady state probabilities are 9% (state 1), 40% (state 2), 28% (state 3) and 23% (state 4). Hence, although the crash state is clearly not visited as often as the other states, it by no means only picks up extremely rare events.

It is interesting to relate these states to the underlying business cycle. Correlations between smoothed state probabilities and NBER recession dates are 0.32 (state 1), -0.13 (state 2), -0.21 (state 3), and 0.18 (state 4). Notice that since the state probabilities sum to one, by construction if some correlations are positive, others must be negative. This suggests that indeed, the high-volatility states - states 1 and 4 - occur around official recession periods.[6]

3. The Investor's Asset Allocation Problem

We next study the asset allocation implications of regime dynamics in the joint distribution of stock and bond returns. First consider the 'pure' asset allocation problem for an investor with power utility defined over terminal wealth, W_{t+T}, coefficient of relative risk aversion $\gamma > 1$, and an investment horizon T:

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma}. \tag{7}$$

[5]Recent work by Andersen, Bollerslev, Diebold, and Vega (2004) reaches the same conclusion: stock and bond returns move together insofar as the correlation is sizeable and important, but it switches sign different regimes, and it therefore may appear spuriously small when averaged across states.

[6]It may be argued that the state probabilities backed out from financial returns should lead economic recession months. Indeed, the correlation between the state 1 probability lagged 6 months and the NBER recession indicator rises to 0.40.

The investor is assumed to maximize expected utility by choosing at time t a portfolio allocation to large stocks, small stocks and bonds, $\boldsymbol{\omega}_t^T \equiv (\omega_t^l(T)\ \omega_t^s(T)\ \omega_t^b(T))'$, while $1-(\boldsymbol{\omega}_t^T)'\boldsymbol{\iota}_3$ is invested in riskless T-bills.[7] For simplicity we assume the investor has unit initial wealth. Portfolio weights are adjusted every $\varphi = \frac{T}{B}$ months at B equally spaced points t, $t+\frac{T}{B}$, $t+2\frac{T}{B}$, ..., $t+(B-1)\frac{T}{B}$. When $B=1$, $\varphi = T$ and the investor simply implements a buy-and-hold strategy.

Let $\boldsymbol{\omega}_b$ $(b = 0, 1, ..., B-1)$ be the portfolio weights on the risky assets at these rebalancing times. Then $1-\boldsymbol{\omega}_b{}'\boldsymbol{\iota}_3$ is the weight on T-bills at time $t + b\frac{T}{B}$ and

$$u(W_B) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma} = \frac{W_B^{1-\gamma}}{1-\gamma}.$$

With regular rebalancing the investor's optimization problem is

$$\begin{aligned} &\max_{\{\boldsymbol{\omega}_j\}_{j=0}^{B-1}} \quad E_t\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right] \\ s.t.\ W_{b+1} &= W_b\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right\} \\ \mathbf{R}_{b+1} &\equiv \mathbf{r}_{t_b+1} + \mathbf{r}_{t_b+2} + ... + \mathbf{r}_{t_{b+1}}, \quad b = 0, 1, ..., B-1. \end{aligned} \tag{8}$$

The equation for the wealth evolution is exact when asset returns are continuously compounded and excess returns are computed as the difference between asset returns and the risk-free rate.[8] Incorporating investors' use of predictor variables $\mathbf{z}_b$, at the decision times $b = 0, 1, ..., B-1$, we get the following derived utility of wealth

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j\}_{j=b}^{B-1}} E_{t_b}\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right]. \tag{9}$$

Here $\boldsymbol{\theta}_b = \left(\left\{\boldsymbol{\mu}_i, \boldsymbol{\mu}_{zi}, \boldsymbol{\Omega}_{i,b}^*, \{\mathbf{A}_{j,i,b}^*\}_{j=1}^p\right\}_{i=1}^k, \mathbf{P}_b\right)$ collects the parameters of the regime switching model and $\boldsymbol{\pi}_b$ is the (column) vector of probabilities for each of the k possible states conditional on information at time t_b. Consistent with common practice (e.g. Aït-Sahalia and Brandt (2001), Brennan, Schwartz, and Lagnado (1997), and Brennan and Xia (2002)), we rule out short-selling. Let $\mathbf{e}_j$ be a 3×1 vector of zeros with a 1 in the jth place and $\boldsymbol{\iota}_3$ be a 3×1 vector of ones. No short sales then means that $\mathbf{e}_j'\boldsymbol{\omega}_b \in [0, 1]$ $(j = 1, 2, 3)$ and $\boldsymbol{\omega}_b'\boldsymbol{\iota}_3 \leq 1$.[9] We also ignore capital gains taxes and other frictions.[10]

[7] Following standard practice we consider a partial equilibrium framework which takes the asset return process as exogeneous (c.f. Ang and Bekaert (2001)) and assume that the risk-free rate is constant and equal to the average 1-month T-bill yield over the sample period (5.3% per year). In the following, unless necessary, we do not explicitly indicate the investment horizon when referring to the vector of portfolio weights $\boldsymbol{\omega}_t^T$.

[8] This is the same equation as in Ang and Bekaert (2001) and Barberis (2000).

[9] Short-selling constraints only have a marginal impact on our results as they are not binding except at the very short investment horizons. This finding is similar to results in Detemple, Garcia, and Rindisbacher (2003). The intuition is that nonlinear processes may imply long-run (ergodic) densities of the data that are far less 'extreme' (in terms of portfolio weights) than those obtained by iterating over long horizons the typical linear-VAR type models of predictable expected returns (e.g. Campbell and Viceira, 1999). As pointed out by Kandel and Stambaugh (1996), the portfolio can go bankrupt if it is fully invested in an asset with a return of -100%. With zero wealth, the investor's objective function becomes unbounded, preventing an interior solution from existing. We use a simple rejection algorithm to ensure that wealth remains positive at all horizons along all simulation paths. This is equivalent to truncating the joint density from which asset returns are drawn. In practice we never found that rejections occurred on the simulated paths.

[10] Dammon, Spatt, and Zhang (2001) analyze the effects of capital gains taxes on optimal consumption and asset allocation

Under power utility the Bellman equation conveniently simplifies to

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) = \frac{W_b^{1-\gamma}}{1-\gamma} Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \quad (\gamma \neq 1). \tag{10}$$

Since the states are unobservable, investors' learning is incorporated in this setup by letting them optimally revise their beliefs about the underlying state at each point in time using the updating equation

$$\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) = \frac{\left(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi}\right)' \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t))]'\boldsymbol{\iota}_k}, \tag{11}$$

where a 'hat' on top of a parameter indicates that it is an estimate, $\odot$ denotes the element-by-element product, $\mathbf{y}_b \equiv (\mathbf{r}_b' \ \mathbf{z}_b')'$, $\hat{\mathbf{P}}_t^{\varphi} \equiv \prod_{i=1}^{\varphi} \hat{\mathbf{P}}_t$, and $\boldsymbol{\eta}(\mathbf{y}_{b+1})$ is the $k \times 1$ vector whose jth element gives the density of observation $\mathbf{y}_{b+1}$ in the jth state at time t_{b+1} conditional on $\hat{\boldsymbol{\theta}}_b$:

$$\boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_b) \equiv \begin{bmatrix} f(\mathbf{y}_{b+1}|s_{b+1} = 1, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \\ f(\mathbf{y}_{b+1}|s_{b+1} = 2, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \\ \vdots \\ f(\mathbf{y}_{b+1}|s_{b+1} = k, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \end{bmatrix}$$

$$= \begin{bmatrix} (2\pi)^{-\frac{N}{2}} |\hat{\boldsymbol{\Omega}}_1^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{1j}\mathbf{y}_{t_b-j}\right)' \hat{\boldsymbol{\Omega}}_1^{-1} \left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{1j}\mathbf{y}_{t_b-j}\right)\right] \\ (2\pi)^{-\frac{N}{2}} |\hat{\boldsymbol{\Omega}}_2^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{2j}\mathbf{y}_{t_b-j}\right)' \hat{\boldsymbol{\Omega}}_2^{-1} \left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{2j}\mathbf{y}_{t_b-j}\right)\right] \\ \vdots \\ (2\pi)^{-\frac{N}{2}} |\hat{\boldsymbol{\Omega}}_k^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{kj}\mathbf{y}_{t_b-j}\right)' \hat{\boldsymbol{\Omega}}_k^{-1} \left(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{kj}\mathbf{y}_{t_b-j}\right)\right] \end{bmatrix} \tag{12}$$

Our approach is consistent with the notion that investors never observe the true state. Learning effects can be important since optimal portfolio choices depend not only on future values of asset returns and predictor variables ($\mathbf{r}_b$, $\mathbf{z}_b$), but also on future perceptions of the likelihood of being in each of the unobservable regimes ($\boldsymbol{\pi}_{t_b+j}$), c.f. Gennotte (1986).

Since W_b is known at time t_b, $Q(\cdot)$ simplifies to

$$Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} E_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q\left(\mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right]. \tag{13}$$

In the absence of predictor variables, $\mathbf{z}_t$, the investor's perception of the regime probabilities, $\boldsymbol{\pi}_b$, is the only state variable and the basic recursions can be written as:

$$\begin{aligned} Q(\boldsymbol{\pi}_b, t_b) &= \max_{\boldsymbol{\omega}_b} E_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right], \\ \boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) &= \frac{(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k}. \end{aligned} \tag{14}$$

decisions when short sales are restricted but asset returns are *not* predictable.

3.1. *Numerical Solutions*

Various approaches have been followed in the literature on portfolio allocation under predictable returns. Barberis (2000) employs simulation methods to study a 'pure' allocation problem without interim consumption. Ang and Bekaert (2002) solve for the optimal asset allocation using Gaussian quadrature methods. Campbell and Viceira (1999, 2001) and Campbell, Chan and Viceira (2003) derive approximate analytical solutions for an infinitely lived investor. Finally, some papers have derived closed-form solutions by working in continuous-time, e.g. Kim and Omberg (1996) for the case without interim consumption and Wachter (2002) for the case with interim consumption and complete markets.

Ang and Bekaert (2002) were the first to study asset allocation under regime switching. They consider pairs of international stock market portfolios under regime switching with observable states, so the state variable simplifies to a set of dummy indicators. This setup allows them to apply quadrature methods based on a discretization scheme. Our framework is quite different since we treat the state as *unobservable* and calculate asset allocations under optimal filtering (11).[11]

To deal with the latent state we use Monte-Carlo methods for integral (expected utility) approximation. For example, for a buy-and-hold investor, we follow Barberis (2000) and approximate the integral in the expected utility functional as follows:

$$\max_{\boldsymbol{\omega}_t} N^{-1} \sum_{n=1}^{N} \left\{ \frac{\left[(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_3)\exp\left(Tr^f\right) + \boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T}(r^f\boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n})\right)\right]^{1-\gamma}}{1-\gamma} \right\}. \tag{15}$$

Here $\boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T}(r^f\boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n})\right)$ is the portfolio return in the n-th Monte Carlo simulation. Each simulated path of portfolio returns is generated using draws from the model (1)-(2) that allow regimes to shift randomly as governed by the transition matrix, $\mathbf{P}$. We use $N = 30,000$ simulations. As pointed out by Detemple, Garcia, and Rindisbacher (2003), numerical schemes based either on grid approximation of partial differential equations or on quadrature discretization of the state space suffer from a dimensionality curse that Monte Carlo simulation methods can help alleviate. This makes Monte Carlo methods particularly suitable to a multivariate problems such as ours. Appendix A and B provide details on the numerical techniques employed in the solutions.

4. **Asset Allocation Results**

As a benchmark we first consider the asset allocation strategy of a buy-and-hold investor who solves the asset allocation problem once, at time t. Brennan and Xia (2002) point out that this is an interesting special case since it corresponds to the problem solved by an investor who has set aside predetermined savings for retirement and commits to a portfolio that maximizes the expected utility from consumption upon retirement. At the end of the section we introduce rebalancing. Following Aït-Sahalia and Brandt (2001) we vary the investment horizon T between six and 120 months in increments of six months. The coefficient of relative risk aversion is initially set at $\gamma = 5$.

[11] Ang and Bekaert (2001) conjecture that when regimes are unobservable, the problem becomes considerably more difficult since - as they correctly point out - all possible sample paths must be considered.

Figure 2 plots the optimal asset allocations each quarter at horizons $T = 6, 24$ and 120 months over the period 1980-1999, when the full sample (smoothed) state probabilities in Figure 1 are employed and the parameters are held fixed at their estimated values from Table 2.[12] At intermediate and long investment horizons, portfolio weights are reasonably stable over time as investors acknowledge that the current state will not last indefinitely. The weight on small stocks fluctuates between zero and 50%, while the weight on large stocks varies between zero and 70% and bond holdings change between zero and 30%. Cash holdings average 20% for a long-term investor and fluctuate between zero and 40% over the sample.

To put the effect of regime switching on optimal asset allocations into perspective, Figure 2 also shows asset holdings under independently and identically distributed (IID) returns where the optimal portfolio weights are constant across investment horizons. We refer to these as the 'myopic weights'. The optimal weights in the myopic portfolio are only non-zero for long-term bonds (70%) and large stocks (30%) and the myopic investor does not hold T-bills.

4.1. *Optimal Asset Allocation in the Four Regimes*

We found in Section 2 that the four regimes identified in the joint distribution of stock and bond returns had economic interpretations as crash, slow growth, bull and recovery states. To better understand the role of these economic states in asset allocation, Figure 3 shows optimal asset allocations starting from each of the states, i.e. $\boldsymbol{\pi} = \mathbf{e}_j$ $(j = 1, 2, 3, 4)$, but allowing for uncertainty about future states due to randomly occurring regime shifts driven by (2).

State 1 is a low return state with little persistence. As the investment horizon (T) grows, investors can be reasonably certain of leaving this state and move to better ones. The weight on stocks is therefore negligible for small T but increases as T grows, producing an upward-sloping curve. Although it is sensible to avoid stocks almost completely at short horizons, the low persistence of regime 1 along with the high probability of switching to the high mean recovery state leads to a rapid increase in the optimal allocation to stocks as the horizon expands. Even so, the optimal allocation to stocks never exceeds 35% when starting from the crash state. The allocation to bonds grows from zero to 30%, while the allocation to T-bills shows the opposite pattern, starting at 100% of the portfolio and declining to 40% at the 10-year horizon.

In the slow growth state (regime 2) the small firm effect is negative and the demand for small stocks is always zero while conversely that for large stocks is very high, starting at 100% at the shortest horizon before declining to a level near two-thirds of the portfolio at horizons longer than six months. The remainder of the portfolio is invested in bonds and T-bills.

The bull state is associated with a sizeable small firm effect and small stocks take up 70% of the portfolio at short horizons before declining to 20% for horizons greater than six months. The reverse pattern is seen for large stocks that start at 30% for short horizons and grow to a level near 50% for horizons longer than six months. Bond and T-bill allocations are close to zero at short horizons, rising to around 10% and 15%, respectively, at long horizons.

Finally, starting from the recovery state, 100% of the portfolio is allocated to small stocks for investors with a short horizon. This proportion declines to 40% for horizons longer than one year, while the allocation to large stocks and bonds rise from zero to 30% as the horizon is extended from one to 12 months. In this

[12] Section 7.3 presents the results of a recursive, out-of-sample asset allocation exercise.

state practically nothing is invested in T-bills.

Overall, we find that the well-known investment advice of increased exposure to stocks the longer the horizon is not robust to how predictability in returns is modeled and may even be more of an exception than the rule.[13] In three of four states the buy-and-hold investor is more cautious about stocks as the investment horizon rises. Although our multivariate regime switching model is not nested in the class of non-linear processes studied by Detemple, Garcia, and Rindisbacher (2003), our finding that the current state variable is important to the optimal portfolio weights is similar to theirs.

4.2. *Uncertainty about the initial State*

Our analysis does not assume that investors always know the underlying state. This is significant since, as shown by Veronesi (1999), uncertainty about the underlying regime is important in understanding asset price dynamics. We next examine the asset allocation implications of uncertainty about the initial state by considering two scenarios. The first assumes that the states have the same probability (25%) while the second scenario assumes steady-state probabilities (9%, 40%, 28% and 23% for states 1-4). The extent to which asset allocations depend on the underlying state beliefs is clear from Figure 4: at least for stocks the sign of the slope of the investment demand at short horizons is opposite in the two scenarios.

These results suggest that investors' perceptions of the current state probability is a key determinant of the relationship between the investment horizon and the optimal asset allocation, and therefore of the degree to which an investor can exploit predictability in asset returns. We discuss these effects in more detail in the next section.

4.3. *Effects of Risk Aversion*

Up to this point we assumed a coefficient of relative risk aversion of $\gamma = 5$, but it is of interest to see how strongly the results vary across different values of this parameter. Starting from each of the four regimes Figure 5 therefore shows portfolio weights as a function of γ. To save space, we focus on the combined allocation to (small and large) stocks and bonds and present plots for three investment horizons, $T = 1, 24,$ and 120 months. For comparison we also show results under the benchmark of no predictability. Independently of the current state, the overall allocation to stocks declines as γ increases. Irrespective of γ, states clearly matter most at the short and medium horizons and their presence leads to a very different allocation from that under IID returns.

Under the IID model the bond allocation rises sharply at low levels of risk aversion and peaks at a level close to 85% for γ just below 20. Far less is allocated to bonds under the regime switching model irrespective of the initial state although, on a much smaller scale, a similar non-monotonic pattern is observed. These patterns are similar to those reported by Aït-Sahalia and Brandt (2001) who find that the allocation to bonds is large (30%) only for $T = 1$ and in good states. Plausible values of γ suggest that asset allocations under regime switching are very different from those in its absence.

[13] Siegel (1994) argues that long-run investors should not try to time the stock market. Our results show that both the proportion allocated to stocks and its decomposition into small and large stocks depends on the initial state. Barberis (2000) and Xia (2001) show that standard investment advice does not apply when parameter uncertainty is taken into account. Our result is different since we obtain non-monotonic investment schedules without introducing parameter estimation uncertainty.

4.4. *Rebalancing*

The buy-and-hold results presented thus far ignore the possibility of rebalancing. However, in the presence of time-varying investment opportunities, investors should adjust their portfolio weights as new information arrives. We therefore next consider the effects of periodic rebalancing on optimal asset allocations. Once again we numerically solve the Bellman equation by discretizing the compact interval that defines the domain of each of the state variables on G points and use backward induction methods. Suppose that $Q(\boldsymbol{\pi}_{b+1}, t_{b+1})$ is known at all points $\boldsymbol{\pi}_{b+1} = \boldsymbol{\pi}^j_{b+1}$, $j = 1, 2, ..., G^{k-1}$. This will be true at time $t_B \equiv t + T$ as $Q(\boldsymbol{\pi}^j_B, t_B) = 1$ for all values of $\boldsymbol{\pi}^j_B$ on the grid. Then we can solve equation (8) to obtain $Q(\boldsymbol{\pi}_b, t_b)$ by choosing $\boldsymbol{\omega}_b$ to maximize

$$E_{t_b}\left[\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1,n}(s_b)+\varphi r^f\boldsymbol{\iota}_3)\right\}^{1-\gamma}Q(\boldsymbol{\pi}^j_{b+1}, t_{b+1})\right]. \tag{16}$$

The multiple integral defining the conditional expectation is again calculated by Monte Carlo methods. For each $\boldsymbol{\pi}_b = \boldsymbol{\pi}^j_b$, $j = 1, 2, ..., G^{k-1}$ on the grid we draw N samples of $\varphi-$period excess returns $\{\mathbf{R}_{b+1,n}(s_b) \equiv \sum_{i=1}^{\varphi} \mathbf{r}_{t_b+i,n}(s_b)\}_{n=1}^{N}$ from the regime switching model and approximate (16) as

$$N^{-1}\sum_{n=1}^{N}\left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1,n}(s_b)+\varphi r^f\boldsymbol{\iota}_3)\right\}^{1-\gamma}Q(\boldsymbol{\pi}^{(j,n)}_{b+1}, t_{b+1})\right]. \tag{17}$$

Here $\boldsymbol{\pi}^{(j,n)}_{b+1}$ denotes the element $\boldsymbol{\pi}^j_{b+1}$ on the grid used to discretize the state space that$-$using the distance measure $\sum_{i=1}^{k-1}|\boldsymbol{\pi}^j_{b+1}e_i - \boldsymbol{\pi}_{b+1,n}e_i|-$is closest to

$$\boldsymbol{\pi}_{b+1,n} = \frac{(\boldsymbol{\pi}_b'\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1,n};\hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}_b'\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1,n};\hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k}.$$

Starting from t_{B-1}, we work backwards through the B rebalancing points until $\boldsymbol{\omega}_t \equiv \boldsymbol{\omega}_0$ is obtained. Appendix B provides further details on the iterative backward solution to the asset allocation problem.

Table 3 shows optimal portfolio weights for stocks and bonds under different values of the rebalancing frequency, $\varphi = 1, 3, 6, 12, 24$ months as well as under the buy-and-hold scenario, $\varphi = T$. For a given investment horizon, T, as φ declines investors become more responsive to the current state probabilities. The smaller is φ, the shorter is the period over which the investor commits wealth to a given portfolio. As a result, the investor puts less weight on the steady-state return distribution and increasing weight on the current state, S_t. Consequently, the weight on stocks in the crash state declines as φ decreases and rebalancing becomes more frequent. For instance, when $T = 120$ and $\varphi = 1$ (monthly rebalancing), investors hold no stocks in the crash state, preferring instead to wait for an improvement in the investment opportunity set. In contrast, when φ exceeds the average duration of this regime (e.g., $\varphi = 12$), it is optimal to invest some money in stocks (40%), although the weight remains quite low. In states 2-4 investors increase their allocation to stocks as the time between rebalancing declines. In fact, when $\varphi = 1$, the optimal weight on stocks is close to 100% in these three regimes, irrespective of the investment horizon. Keeping the rebalancing frequency, φ, constant, the demand for stocks is mostly upward sloping although increasingly flat as φ declines. Once again, we find that it is not generally true that investors with longer horizons should allocate more to stocks.

As the investment horizon grows, non-monotonic patterns are observed in the allocation to bonds which in most cases first rises and then declines. Starting from the crash state the allocation to bonds is generally lower, the more frequent the rebalancing (smaller φ) since the investor does not have to account for unexpected shifts to a better state but can afford to wait for such a shift to occur. If rebalancing can occur frequently, little or nothing is invested in bonds since market timing opportunities are more significant for stocks and the remainder can be held in T-bills.

5. **Asset Allocation Under Predictability from the Dividend Yield**

A large literature in finance has reported evidence that variables related to the business cycle predict stock and bond returns. One of the key instruments is the dividend yield; see, e.g., Campbell and Shiller (1988), Fama and French (1988, 1989), Ferson and Harvey (1991), Goetzmann and Jorion (1993) and Kandel and Stambaugh (1996). Due to its high persistence coupled with the strong negative correlation between shocks to returns and shocks to the dividend yield, Campbell, Chan, and Viceira (2003) find that the dividend yield generates the largest hedging demand among a wider set of predictor variables.

5.1. *Allocations under a Single State Model*

Asset allocation implications of linear predictability in returns from variables such as the dividend yield have been considered by Barberis (2000), Campbell and Viceira (1999), Campbell, Chan, and Viceira (2003), Kandel and Stambaugh (1996) and Xia (2001). It is therefore natural to compare our results to those arising from a standard VAR(1) model comprising asset returns and the dividend yield:

$$\begin{pmatrix} \mathbf{r}_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}_{dy} \end{pmatrix} + \mathbf{A} \begin{pmatrix} \mathbf{r}_{t-1} \\ dy_{t-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \varepsilon_{dy,t} \end{pmatrix}. \tag{18}$$

where $\mathbf{r}_t \equiv (r_t^l \; r_t^s \; r_t^b)'$ and $(\boldsymbol{\varepsilon}_t' \; \varepsilon_{dy,t})' \sim N(0, \Omega)$. MLE estimates are as follows (standard errors are in parentheses below the point estimates):

$$\begin{pmatrix} \mathbf{r}_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \underset{(0.0070)}{0.0021} \\ \underset{(0.0102)}{-0.0160} \\ \underset{(0.0036)}{-0.0032} \\ \underset{(0.0003)}{0.0004} \end{pmatrix} + \begin{bmatrix} \underset{(0.0635)}{-0.0466} & \underset{(0.0412)}{0.0370} & \underset{(0.0839)}{0.2299} & \underset{(0.2028)}{0.1261} \\ \underset{(0.0925)}{0.1236} & \underset{(0.0600)}{0.1244} & \underset{(0.1233)}{0.2624} & \underset{(0.2953)}{0.6641} \\ \underset{(0.0330)}{-0.0442} & \underset{(0.0214)}{-0.0261} & \underset{(0.0436)}{0.1070} & \underset{(0.1054)}{0.1322} \\ \underset{(0.0024)}{-0.0005} & \underset{(0.0016)}{-0.0005} & \underset{(0.0032)}{-0.0098} & \underset{(0.0077)}{0.9856} \end{bmatrix} \begin{pmatrix} \mathbf{r}_{t-1} \\ dy_{t-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \varepsilon_{dy,t} \end{pmatrix},$$

where $r_t \equiv [r_t^l \; r_t^s \; r_t^b]'$. The estimated covariance matrix is:[14]

$$\hat{\boldsymbol{\Omega}}^* = \begin{bmatrix} 0.1417^{***} & 0.0018 & 0.0002 & -5.86e^{-05} \\ 0.7285^{***} & 0.2063^{***} & 0.0002 & -7.10e^{-05} \\ 0.2466^{*} & 0.1353 & 0.0736^{***} & -7.95e^{-06} \\ -0.9243^{***} & -0.7695^{***} & -0.2413 & 0.0056^{***} \end{bmatrix}.$$

[14]Below diagonal coefficients are implied correlation coefficients. * denotes significance at the 10% level, ** at 5%, and *** at 1%. As in Table 2 we report annualized volatilities on the main diagonal.

The estimate of $\hat{\mathbf{A}}^*$ suggests that a higher dividend yield forecasts higher asset returns. The dividend yield is highly persistent - its autoregressive coefficient estimate is almost 0.99 - and shocks to the dividend yield are highly negatively correlated with shocks to stock returns (-0.92 and -0.76 for large and small stocks, respectively), suggesting that time-variations in the dividend yield may induce a large hedging demand for stocks. In contrast, shocks to the dividend yield are only mildly−and insignificantly−correlated with shocks to bond returns (-0.24).[15]

Figure 6 reports the allocations to stocks and bonds under the VAR(1) for a range of values of the dividend yield. Our results are comparable to earlier findings: at most values of the dividend yield the overall allocation to stocks is larger, the longer the investment horizon or the higher the initial value of the dividend yield. The emergence of some slight non-monotonic investment schedules for stocks under the VAR(1) model is not completely surprising. Authors such as Aït-Sahalia and Brandt (2001), Brandt (1999) and Barberis (2000) have found increasing equity demand, the longer the investment horizon when conditioning on a value of the dividend yield close to its sample average. However, when the dividend yield is further away from its unconditional mean, asset allocation results become more mixed and there are cases where, at short-to-intermediate investment horizons, the equity demand is declining in the horizon.

Figure 6 also reveals some interesting substitution effects between small and large stocks. When the dividend yield is very high or high (5.7% or 4.6%), the short-term allocation to small stocks is very large while conversely large stocks are mostly excluded from the portfolio. As the investment horizon grows, the allocation to small stocks declines while that of the large stocks increases by an equivalent amount. More generally, medium-high dividend yields favor small stocks while medium-low yields increase the demand for large stocks. The reason for this is the greater sensitivity of the small stocks' returns to the dividend yield (0.66) compared with the sensitivity of the large stocks (0.12).

There is very little role for bonds in the optimal asset allocation under a VAR(1) model. This holds across all initial values of the dividend yield. The reason is seen in the plot for the allocation to T-bills. When the dividend yield is either low or very low - so stocks are unattractive - short-term investors respond not by holding a larger proportion of bonds, but rather by increasing their allocation to T-bills.

5.2. *Regimes and Predictability from the Dividend Yield*

We next investigate the effect of adding the dividend yield to our model. The resulting regime switching VAR model nests many of the models in the existing literature and enables the correlation between the dividend yield and asset returns to vary across different regimes. The relationship between stock returns and the dividend yield is linear within a given regime. However, since the coefficient on the dividend yield varies across regimes, as the regime probabilities change the model is capable of tracking a non-linear relationship between asset returns and the yield. This is important given the evidence suggesting a non-linear relationship between the yield and stock returns uncovered by Ang and Bekaert (2004).

[15] Our choice of an unrestricted VAR(1) model is consistent with Campbell, Chan, and Viceira (2003). In strategic asset allocation problems involving investments in bonds it is important to allow for predictability from lagged bond returns to current stock returns and the zero restrictions on the VAR(1) return coefficients are strongly rejected by a likelihood ratio test. Following studies such as Barberis (2000) and Lynch (2001) we also considered a restricted VAR(1) model that sets the coefficients of lagged returns equal to zero. This is equivalent to simply 'turning off' the first three columns of $\hat{\mathbf{A}}$. Results were qualitatively similar to those reported in Figure VI.

Again we conducted a battery of tests to determine the best model specification. The results, shown in Panel B of Table 1, suggest that a four-state VAR(1) model is supported by the data as this model passes all diagnostic tests.[16] Unsurprisingly, given the persistence in the dividend yield, a single lag is required for this extended model. Regime 1 continues to be characterized by large, negative mean excess returns. The dividend yield is relatively high in this state (4%) and volatility is also above average. In steady state this regime occurs 15% of the time although it has an average duration of only two months. Regime 2 remains a slow growth state with moderate volatility. This state is highly persistent, lasting on average almost 16 months and occurring close to one-third of the time. Regime 3 continues to be a highly persistent bull state that lasts on average almost 15 months. Finally, regime 4 is again a recovery state with strong stock market rallies accompanied by substantial volatility. This state has an average duration of only two months. Nevertheless, at 18%, its steady-state probability is quite high.

To study the asset allocation effects of regimes and predictability from the yield we report two exercises. The first, presented in Figure 7, shows the optimal asset allocation as a function of the investment horizon when the dividend yield is set at its overall sample average. Asset allocations continue to vary significantly across the four states. Starting from state 1 the allocation to stocks (small stocks in particular) continues to rise as a function of the horizon and peaks at close to 40% of the portfolio at the 10-year horizon. The allocation to bonds is non-monotonic, starting from zero at the shortest horizon, rising to a level close to 90% at the six month horizon before declining to 60% at the longest horizon. T-bills form 100% of the portfolio at the shortest 1-month horizon but then see their allocation decline sharply to zero at horizons longer than six months.

The allocation to stocks continues to decline when the model starts from states two or four, although it only declines to a level near 80-85% at the 10-year horizon. The allocation to bonds makes up for the remainder and there is no demand for T-bills in these two states. In the third (bull) state the allocation to stocks is now mildly upward sloping as a function of the horizon in contrast to what we found in the model without the dividend yield shown in Figure 3.

Figure 8 shows the effect of changing the dividend yield using a range of values spanning its mean value plus or minus three standard deviations. As expected, the higher the dividend yield, the larger the allocation to stocks. This is consistent with the common finding of a positive correlation between the yield and expected returns. The allocation to small stocks is more sensitive to the yield than that of the large stocks. When the yield is very low, the allocation to stocks is very small and the allocation to T-bills is large, but it declines as a function of the investment horizon. Irrespective of the presence of regimes, we get very sensible results for the effect of changing the dividend yield on the optimal asset allocation.

We summarize these findings as follows. First, by comparing Figures 3 and 7, it is obvious that the dividend yield continues to have an important effect on the optimal asset allocation even in the presence of regimes. In the model extended to include the yield there is less of a role for T-bills, while conversely long-term bonds and large stocks form a larger part of the portfolio. Furthermore, irrespective of the presence of regimes, the higher the yield, the greater the typical allocation to stocks.

[16] To save space we do not report parameter estimates for the extended model, but results are available on request.

5.3. *Model Comparison*

Several conclusions emerge from our analysis so far. First, in the pure regime switching model the buy-and-hold investor's allocation to stocks declines as a function of the investment horizon in three out of four states (Figure 3)−and also when the initial state probabilities are set at their steady-state values−and only increases when starting from the crash state. In the single-state VAR(1) model the allocation to stocks is either upward-sloping or constant as a function of the investment horizon (Figure 6). Finally, the allocation to stocks is upward sloping for four out of six configurations of initial state probabilities in the regime switching model extended to include predictability of returns from the dividend yield (Figure 7).

To explain these findings, consider the following simple two-period example that decomposes returns on a risky asset, r_{t+1}, into an expected component, $E_t[r_{t+1}]$, and an innovation, u_{t+1}:[17]

$$r_{t+1} = E_t[r_{t+1}] + u_{t+1}, \tag{19}$$

where $Var_t(r_{t+1}) = \sigma_u^2$. At the two-period horizon, cumulated returns become

$$r_{t+1} + r_{t+2} = E_t[r_{t+1}] + u_{t+1} + E_{t+1}[r_{t+2}] + u_{t+2},$$

and so the variance of two-period returns is

$$Var(r_{t+1} + r_{t+2}) = 2\sigma_u^2 + Var(E_{t+1}[r_{t+2}]) + 2Cov(u_{t+1}, E_{t+1}[r_{t+2}]).$$

Comparing single-period and two-period return variances, we have

$$\frac{Var(r_{t+1} + r_{t+2})}{2Var(r_{t+1})} = 1 + \frac{1}{2}\left(\frac{R^2}{1-R^2}\right)\beta, \tag{20}$$

where

$$\begin{aligned} R^2 &= \frac{Var(E_t[r_{t+1}])}{Var(E_t[r_{t+1}]) + \sigma_u^2}, \\ \beta &= \frac{Cov(u_{t+1}, E_{t+1}[r_{t+2}])}{\sigma_u^2}. \end{aligned}$$

Models of learning (e.g. Brennan (1998)) where investors revise upwards their expectations of future returns following positive return shocks imply that $\beta > 0$. For instance, in our pure regime switching model positive shocks in the future will induce belief revisions in favor of states with high expected returns. Notice that our results under rebalancing incorporate the effects of learning , since in this case we simulate not only returns given initial state beliefs, but also the updating of such beliefs at the rebalancing points. From (20) this means that the variance of two-period returns exceeds twice the variance of the single-period return, suggesting that risk grows faster than at the rate implied by the constant expected return model ($\beta = 0$). Since the investment demand is independent of the horizon under the constant expected return model, such learning effects tend to lead to a demand for the risky asset that declines in the investment horizon.

Conversely, models of predictable, mean-reverting returns imply $\beta < 0$. For example, a negative shock to returns in the VAR(1) model implies a higher value of the dividend yield and hence higher expected

[17] We are grateful to John Campbell for suggesting this analysis.

future returns. Hence the risk of stock returns grows at a slower rate than if expected returns were constant. This tends to lead to an increased demand for the risky asset, the longer the horizon.

This simple analysis provides intuition for our basic findings. The analysis is of course complicated by the fact that under regime switching both the mean and variance of returns depend on the state probabilities. The exception to this is when the state probabilities are set at their steady-state values in which case expected returns become independent of the investment horizon. For this scenario Figure 9 plots the volatility and Sharpe ratio implied by the models considered so far. We show results for small and large stocks only as the effects are much smaller for bonds. The volatility and Sharpe ratio are normalized by dividing by $\sqrt{T}$ so the benchmark IID model corresponds to a straight line.

First consider the pure regime switching model. Starting from the steady state probabilities the mean return is constant whereas the volatility per month increases as a function of the investment horizon. This leads to a Sharpe ratio that declines in the investment horizon and hence to a lower allocation to risky assets. Consistent with this, Figure 3 showed that it is only when the model starts from the crash state that the overall allocation to stocks is increasing in the horizon−the reason being that the mean return increases as a function of the investment horizon while the risk declines when starting from this state.

Next consider the VAR(1) model where the initial dividend yield is set at its unconditional mean. For this model Figure 9 shows that the volatility decreases and hence the Sharpe ratio increases (relative to the IID benchmark) as a function of the investment horizon, leading to a greater allocation to stocks the longer the investment horizon, as we found in Figure 6.

In the four-state model extended to include the dividend yield, learning effects−which tend to lower the allocation to stocks ($\beta > 0$)−compete with mean reversion effects, which tend to increase the allocation to stocks ($\beta < 0$) the longer the investment horizon. Which effect dominates is an empirical issue that also depends on the initial values assumed for the dividend yield and the state probabilities. In practice, it seems that learning effects are stronger at short horizons, so that Sharpe ratios tend to decrease in T. However, at horizons in excess of one or two years, learning effects become weaker as the predictive distribution of returns converges to the steady state distribution, so mean-reversion effects captured by linear predictability from the dividend yield eventually lead to increasing Sharpe ratios.

6. **Optimal Savings and Portfolio Choices**

Some papers (e.g., Brandt (1999), Campbell and Viceira (1999) and Campbell, Chan, and Viceira (2003)) have considered interim consumption and we follow this literature by letting the investor consume and rebalance portfolio weights every φ months. Let $\boldsymbol{\omega}_b$ and C_b be the portfolio allocations and consumption flow at the rebalancing times $t + b\frac{T-t}{B}$, $b = 0, 1, ..., B-1$. Assuming the investor has additively separable power utility, expected lifetime utility is given by

$$E_t\left[\sum_{b=0}^{B} \beta^b \frac{C_{t+b}^{1-\gamma}}{1-\gamma}\right], \tag{21}$$

where the expectation is taken conditional on information available at time t ($b = 0$) and the subjective discount factor is $\beta = (1+\rho)^{-\frac{\varphi}{12}}$, where $\rho = 0.05$ is the investor's annualized subjective rate of time preference.[18] In the presence of consumption, the investor's wealth follows the process

$$\begin{aligned} W_{b+1} &= (W_b - C_b)\left[(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right] \\ &= W_b\psi_b\left[(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right], \end{aligned} \tag{22}$$

where $\psi_b \equiv (W_b - C_b)/W_b$ is the fraction of wealth saved at time $t+b$ and $\mathbf{R}_{b+1} \equiv \mathbf{r}_{t_b+1} + \mathbf{r}_{t_b+2} + ... + \mathbf{r}_{t_{b+1}}$. The dynamic program implied by (21)-(22) can be solved by choosing a sequence of portfolio allocations and savings ratios $\{\boldsymbol{\omega}_b, \psi_b\}_{b=0}^{B-1}$ to get the following value function:

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j,\psi_j\}_{j=b}^{B-1}} E_{t_b}\left[\sum_{j=0}^{B}\beta^j\frac{(1-\psi_{b+j})^{1-\gamma}W_{b+j}^{1-\gamma}}{1-\gamma}\right]. \tag{23}$$

The Bellman equation is

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\boldsymbol{\omega}_b,\psi_b}\left\{\frac{(1-\psi_b)^{1-\gamma}W_b^{1-\gamma}}{1-\gamma} + \beta E_{t_b}\left[J(W_{b+1}, \mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\theta}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1})\right]\right\},$$

subject to the constraint in (22). Under power utility this simplifies to

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) = \frac{W_b^{1-\gamma}}{1-\gamma}Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b).$$

When $\gamma \neq 1$ both ψ_b and $\boldsymbol{\omega}_b$ must be determined using numerical methods and ψ_b depends on the optimal portfolio weights.

Assuming once again that the investor uses the optimal filtering algorithm (11) to update state probabilities, $\hat{\boldsymbol{\pi}}_t$, in the absence of predictor variables the problem simplifies to the basic recursions

$$\begin{aligned} Q\left(\boldsymbol{\pi}_b, t_b\right) &= \max_{\boldsymbol{\omega}_b,\psi_b}\Big\{(1-\psi_b)^{1-\gamma} + \beta\psi_b^{1-\gamma} \\ &\quad \times E_{t_b}\left[\left((1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right]\Big\} \\ \boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) &= \frac{(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}_b'(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k}. \end{aligned}$$

Again, we discretize the state space and solve the Bellman equation by backward induction. Appendix A provides further details.

Figure 10 shows how the consumption-wealth ratio depends on the underlying state probabilities and the investment horizon by plotting C_t/W_t in each of the four regimes assuming consumption only takes place at the beginning and at the end of the investment horizon; like in Brandt (1999). For comparison we also show the optimal value of C_t/W_t obtained under steady-state probabilities.[19] At short horizons all consumption schedules start at roughly 50% so half of the wealth is consumed.

[18] This is the value employed by Brandt (1999). We considered alternative values of ρ and found that our qualitative conclusions on the effects of regimes on consumption were intact.

[19] The portfolio weights from this joint consumption-asset allocation problem coincide with those reported in Section 4.1 since optimal portfolio decisions do not depend on future savings decisions (c.f. Ingersoll (1987, pp. 240-242)). Therefore, the asset allocation results are omitted. Notice, however, that the converse result does not hold, as consumption decisions depend on future investment opportunities.

In the crash state, short-term investment opportunities are poor so the optimal consumption schedule is upward sloping, albeit rather flat and uniformly below that starting from the steady-state probabilities. Even at a 10-year horizon, no more than 55% of wealth should be consumed initially when starting from the crash state. In contrast, investment opportunities are both good and persistent in state 3, so strong income effects induce a rational investor to consume a higher percentage of wealth—as high as 67% for a 10-year horizon. Although investment opportunities are very good in state 4, they are also transitory and switches to the crash state (state 1) have a high probability. Uncertainty about future states accounts for the rather flat consumption demand schedule that initiates from this state.

Overall, these results are consistent with those reported by Brandt (1999) and Campbell and Viceira (1999). In a two-period buy-and-hold exercise, Brandt finds that consumption choices are insensitive to the values assumed by a range of prediction variables and do not differ systematically from their unconditional estimates. Numerically, his estimates of the optimal consumption-wealth ratio are close to ours, the main difference being that we find a stronger sensitivity of C_t/W_t to the horizon in state three. The likely explanation for this is that the prediction variables used in Brandt's study mostly pick up low-frequency predictability patterns (e.g., the dividend yield typically changes very slowly) while our regime switching model captures predictability at a higher frequency that affects not just the conditional mean but the entire probability distribution of returns. Campbell and Viceira (1999) report in an infinite horizon framework that the optimal consumption-wealth ratio mostly depends on preferences (γ and β in our set up) and is only moderately affected by predictor variables such as the dividend yield.

To allow for a more realistic setting with interim consumption over the interval $[t, T]$ Table 4 presents results for $\varphi = 3, 12$, and T. Once again we show separate results starting from each of the four states and also report consumption-wealth ratios calculated under the assumption of logarithmic utility ($\gamma = 1$), when $C_b/W_b = (1 - \beta_\varphi)/(1 - \beta_\varphi^{B-b+1})$ and $\beta_\varphi = (1 + \rho)^{-\frac{\varphi}{12}}$. Substitution and income effects cancel out under this benchmark. Note that when $T \geq \varphi$, a constant consumption rate corresponds to a value of (T/φ) +1, while this rate is simply $T/2$ for values $T < \varphi$.

Under annual consumption and rebalancing ($\varphi = 12$) there is a clear pattern across regimes. In good states (2-4) the consumption-to-wealth ratio is systematically higher than under the logarithmic benchmark, while it is lower in the bad state (state 1). This reflects consumption smoothing: in good states, higher expected portfolio returns can support a higher consumption ratio than if low future returns are anticipated. C_t/W_t is therefore generally higher than under logarithmic utility in states 2-4 and lower than this benchmark in state 1. Allowing for frequent rebalancing, investors reduce current consumption, invest as much as 99% of current wealth and collect the high payoffs/consumption streams later on.

7. **Economic Importance of Regimes**

7.1. *Utility Cost Calculations*

It is natural to report a measure of the economic value of accounting for regimes in investors' asset allocation decisions. Following similar experiments in Ang and Bekaert (2002), Ang and Chen (2002) and Brennan and Xia (2002), we obtain an estimate by comparing the investor's expected utility under the regime switching model to that assuming the investor is constrained to choose at time t an optimal savings ratio ψ_t^{IID} and portfolio weights $\boldsymbol{\omega}_t^{IID}$ under the assumption that asset returns follow a simple IID process. In the latter case the portfolio choice and savings ratio are independent of the investment horizon and the value function for the constrained investor is

$$\begin{aligned} J_t^{IID} &\equiv \frac{1}{1-\gamma}(1-\psi_t^{IID})^{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[W_b^{1-\gamma}\right] \\ W_b &= W_{b-1}\psi_t^{IID}\left[(1-(\boldsymbol{\omega}_t^{IID})'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1}+\varphi r^f\boldsymbol{\iota}_3\right)\right]. \end{aligned} \tag{24}$$

The assumption of IID returns is a constrained special case of the model with regime switching, so

$$J_t^{IID} \leq J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\pi}_t, t),$$

where $J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\pi}_t, t)$ is the value function for the four-state model. We compute the compensatory premium, η_t^{IID}, an investor would be willing to pay to obtain the same expected utility from the constrained and unconstrained consumption and asset allocation problems:

$$\eta_t^{IID} = \left\{\frac{Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b)}{(1-\psi_t^{IID})^{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[(W_b)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{25}$$

Figure 11 plots the annualized riskless compensating rate, $100 \times [\left(1+\eta_t^{IID}\right)^{\frac{12}{T}} - 1]$, needed to make a buy-and-hold investor indifferent between implementing portfolio strategies that exploit the presence of regimes and using the IID portfolio when the current regime probabilities are set at their steady-state values. The utility cost of ignoring regimes is as high as 3% at short horizons−where investors can exploit market timing more aggressively−while, at the longest horizons, the compensating rate is around 130 basis points per annum. This estimate is somewhat larger than the corresponding figures reported by Ang and Bekaert (2002) (20-30 basis points in a regime switching model) but well below the numbers reported by Brennan and Xia (2002) in a model with predictability from inflation and interest rates.

7.2. *Parameter Uncertainty*

The presence of four regimes complicates parameter estimation so we next consider the effect of parameter estimation errors on our results. Since we use Monte Carlo methods to derive optimal portfolio weights, instead of using the delta method as in Ang and Bekaert (2002) and Guidolin and Timmermann (2004b) we exploit that, in large samples,

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right) \stackrel{A}{\to} N(\mathbf{0}, \mathbf{V}_\theta).$$

This allows us to set up the following bootstrap procedure. At step, q, we draw a vector of parameters, $\widehat{\boldsymbol{\theta}}^q$, from $N(\widehat{\boldsymbol{\theta}}, T^{-1}\hat{\mathbf{V}}_\theta)$ where $\hat{\mathbf{V}}_\theta$ is a consistent estimator of $\mathbf{V}_\theta$. Conditional on this draw, $\widehat{\boldsymbol{\theta}}^q$, we solve (8) to obtain a new vector of portfolio weights $\widehat{\boldsymbol{\omega}}^q$. We repeat this process Q times. Confidence intervals for the optimal asset allocation $\hat{\boldsymbol{\omega}}_t$ can then be derived from the distribution for $\widehat{\boldsymbol{\omega}}^q$, $q = 1, 2, ..., Q$. This approach is computationally intensive, as (8) must be solved numerically so we restrict the number of bootstrap trials to $Q = 1,000$. Table 5 shows the optimal asset allocation plus or minus one standard deviation of the bootstrapped distribution. Figures in bold indicate that this band does not include the IID asset allocation, which represents a formal test of the difference between portfolio weights with and without regimes. Standard error bands are wide, but sufficiently narrow to confirm the validity of our conclusions concerning the optimal shape of equity investment schedules as a function of the investment horizon. The allocation to stocks is upward sloping only in the crash state. In regimes 2-4, however, the equity investment schedules are downward sloping, as their bands decline from a maximum of [0.7, 1] at $T = 1$ to [0.4, 0.8] at long investment horizons.

These methods also allow us to consider the joint effect of parameter estimation uncertainty and uncertainty about the underlying state on the utility cost. We do so by calculating the compensating variation $\eta_t^{IID,q}$ 1,000 times using parameter estimates $\widehat{\boldsymbol{\theta}}^q$ drawn from their asymptotic distribution. Figure 12 shows confidence intervals under steady state probabilities. The null hypothesis of zero welfare loss implies that such intervals should include zero for all Ts. Evidence that the lower bound of the interval is positive suggests that ignoring regime switching in asset allocation problems leads to a significant reduction in expected utility.

The null of no significant welfare cost from ignoring regime switching is strongly rejected. The lower bound of the confidence band is everywhere positive and also economically significant. At longer horizons the lower bound attains levels of 7-8%, which is a considerable fraction of wealth. Using a misspecified model in asset allocation decisions may thus be quite costly.

7.3. *Out-of-sample Performance*

A legitimate concern about the results so far is that although the regime switching model leads to sensible portfolio choice recommendations at the end of our sample, it may be difficult to use in 'real time' due to parameter estimation errors which could translate into implausible time-variations in the portfolio weights. This concern is related to the prediction model's out-of-sample asset allocation performance, an issue that has been addressed by authors such as Brennan, Schwartz, and Lagnado (1997), Campbell, Chan, and Viceira (2003), Detemple, Garcia, and Rindisbacher (2003) and Xia (2001).

To get a sufficiently long sample, we first perform a 'pseudo' real time asset allocation exercise for the period 1980:01-1999:12, a total of 240 months. To make the experiment feasible, we focus on the buy-and-hold asset allocation problem at three horizons, $T = 1$, 12, and 120 months. We compare the performance of a four-state regime switching model, the VAR(1) model (18), a four-state regime switching model that includes predictability from the dividend yield, and a simple IID model with constant means and covariance matrix. As additional benchmarks, we also report results for a minimum-variance portfolio and a static, mean-variance tangency portfolio.[20] We preclude the investor from having any benefit of hindsight (c.f. Pesaran and Timmermann (1995)). For instance, a four-state regime switching model is estimated for 1954:01-1979:12 and the estimates and state probabilities as of 1979:12 are used to calculate portfolio performance for 1980:01. Next period the sample is extended to 1954:01-1980:01 and estimation and portfolio optimization is repeated, and so forth.

Interestingly, the turnover in the equity portfolio was found to be smaller under pure regime switching than under the VAR(1) model. Once the dividend yield is included in the regime switching model, the volatility of the equity weights increases and becomes comparable to that under the VAR(1) benchmark. Regime switching increases the overall demand for stocks (approximately 60%) relative to the benchmark

[20]Minimum-variance and tangency portfolios are calculated using sample moments for $T-$period returns. For $T = 120$ we use overlapping returns to have enough sample observations to be able to calculate the required moments.

VAR(1) model (40%) because the models that include the dividend yield as a predictor shift out of stocks during parts of the 1990s. Regimes also have a strong effect on the average demand for small stocks. The weight on these stocks is approximately 40% under regime switching, less than 25% when both regimes and the dividend yield are included, and only 10% under the VAR(1) model.[21] Bonds receive a substantial weight under regime switching−between 35% and 60%, depending on T and irrespective of whether the dividend yield is included as a predictor. Conversely the VAR(1) model puts a large weight on cash investments (in excess of 50%). This suggests that the presence of regimes is important in understanding the demand for (nominal) long-term bonds.

We next calculated realized utility under the different models, each associated with a particular portfolio weight $\hat{\boldsymbol{\omega}}_t^T$ and hence a different realized utility:

$$V_t^T \equiv \frac{\left[W_T(\hat{\boldsymbol{\omega}}_t^T)\right]^{1-\gamma}}{1-\gamma} = \frac{\left[(1-(\hat{\boldsymbol{\omega}}_t^T)'\boldsymbol{\iota}_3)\exp\left(Tr^f\right) + (\hat{\boldsymbol{\omega}}_t^T)'\exp\left(\sum_{j=1}^T \mathbf{r}_{t+j} + Tr^f\boldsymbol{\iota}_3\right)\right]^{1-\gamma}}{1-\gamma}.$$

Here $\gamma = 5$, $T = 1, 12$ and 120 months and $\{\mathbf{r}_{t+\tau}\}_{\tau=1}^T$ are the realized excess asset returns between $t+1$ and $t+T$. The period-t weights, $\hat{\boldsymbol{\omega}}_t^T$, are computed by maximizing the objective $E_t[W_T^{1-\gamma}/1-\gamma]$ so that for each investment horizon, T, and each asset allocation model we obtain a time series $\{V_\tau^T\}$, τ =1980:01,...,1999:12-T of realized utilities. Panels A and B of Table 6 reports summary statistics for the distribution of $\{-V_\tau^T\}$ with smaller values indicating higher welfare. Following Guidolin and Timmermann (2004a), we use a block bootstrap for the empirical distribution of $-V_\tau^T$ to account for the fact that realized utility levels are likely to be serially dependent as time-variations in the conditional distribution of asset returns may translate into dependencies in the portfolio weights and hence in realized utilities.[22]

The VAR(1) model performs best over the shortest investment horizon ($T = 1$) although the 5% and 10% confidence intervals for the realized utility overlap under the VAR(1) and regime switching models suggesting that their performances are statistically indistinguishable. For the longer horizons, $T = 12$, 120 months, the pure regime switching model produces the highest mean realized utility. At the twelve month horizon the out-of-sample forecasting performance of this model is sufficiently good to be statistically significant against three of the five alternative models.

7.3.1. **Performance during 2000-2003**

Although our pseudo out-of-sample results for the period 1980-1999 do not use any data for parameter estimation that was unavailable at the time of the forecast, the choice of model specification could itself have benefited from full-sample information that only became available in 1999. To address this concern

[21]Once the dividend yield is included as a predictor, the demand for stocks is close to zero between 1993 and 1997. This is consistent with the real time results reported by Campbell and Viceira (1999, 2001) and Xia (2001) and is explained by the low value of the dividend yield after 1993 (less than 2.5% vs. an unconditional sample mean of 3.4%). Aït-Sahalia and Brandt (2001, p. 1348) also notice that "(...) in the second half of the 1990s (...) the dividend-to-price ratio stubbornly predicted negative returns for the stock market which never materialized."

[22]The simulation procedure consists of three steps and is separately implemented for each of the horizons $T = 1, 12, 120$ months: 1) Generate B block boostrap resamples of the sample indices $\{1, 2, ..., H\}$ where H is the length of the out-of-sample period. The block length L is set equal to the investment horizon, T. 2) For each of the resamples calculate the summary statistics of interest. 3) Compute bootstrap resample means of the statistics of interest. Our bootstrap implementation uses 50,000 draws.

and to see how the various models performed during 2000-2003, we compute asset allocations and realized utilities over this post-sample period.

Results from this experiment are reported in Panels C and D of Table 6. All models generally suggest a more cautious asset allocation over this period, as reflected by an increase in the demand for T-bills and bonds. At the shortest investment horizon ($T = 1$) the myopic IID, VAR(1) and regime switching model extended by the dividend yield produce almost identical realized utilities. At the intermediate ($T = 12$) horizon, the pure regime switching model performs somewhat worse due to its continued high investment in stocks (70% on average) which stands in contrast to the models that include the dividend yield as a regressor. Since the dividend yield was below its unconditional mean during this sample, both the VAR(1) and regime switching model with the yield included lead to far smaller portions (less than 20%) invested in stocks over this period.

Viewed over the entire out-of-sample period 1980-2003 − and hence averaging across the lengthy bull and bear states of recent years − the four-state model continues to produce the best average realized utility performance at the 1-month and 120-month horizons, while the VAR(1) model generates the best out-of-sample results for the interim 12-month horizon.

8. **Conclusion**

This paper explored the asset allocation implications of the presence of regimes in the joint distribution of stock and bond returns. Our model captures predictability not just in the conditional mean of returns (which most of the existing literature has focused on) but in the full (joint) return distribution, including the volatility, skew and degree of fat-tails. While two states were transitory (the crash and recovery state), the slow growth and bull state are persistent with average durations of several months. This means that the regime switching model captures both short-term and long-term variations in investment opportunities.

We found that the optimal asset allocation varies significantly across regimes as the weights on the various asset classes strongly depend on which state the economy is perceived to be in. Asset allocations therefore vary significantly over time even in the absence of 'outside' predictor variables such as the dividend yield. Stock allocations were found to be monotonically increasing as the investment horizon gets longer in only one of the four regimes. In the other regimes we observed a downward sloping allocation to stocks. The common investment advice of allocating more money to stocks the longer the investment horizon should therefore be made conditional on the underlying state. The extension of the model to a joint consumption and asset allocation problem reveals that the effects of regime switching on consumption decisions is smaller than those on asset allocations, but important nonetheless.

Our framework can be extended in several ways. Aït-Sahalia and Brandt (2001) have experimented with preferences that depart from the standard expected utility, isoelastic benchmark, e.g. calculating optimal portfolio weights under Epstein-Wang ambiguity aversion or loss aversion preferences. Campbell and Viceira (1999) and Campbell et al. (2001) consider Epstein-Zin preferences that disentangle the effects of risk aversion and the elasticity of intertemporal substitution. Our paper evaluates the effects on the optimal asset allocation of modeling asset returns as a data-driven mixture of distributions that can vary significantly across regimes. In related research Brennan and Xia (2001) and Pástor (2000) propose a Bayesian framework to address optimal portfolio choice when investors face model uncertainty and asset return distributions take the form of mixtures over a range of theoretical and data-driven models. Such extensions are likely to deepen our understanding of the effects of multiple regimes on strategic asset allocation and provide interesting challenges for future research. Finally, one could extend our framework to jointly model regimes in equity and bond returns as well as in short-term interest rates which are well-known to incorporate strong non-linearities (see e.g. Ang and Bekaert (2002), Campbell and Yogo (2003) and Garcia et al. (2003)).

Appendix A - Backward Solution of the Joint Consumption and Asset Allocation Problem under Regime Switching

Suppose the optimization problem has been solved backwards at the rebalancing points t_{B-1}, ..., t_{b+1} so that $Q(\boldsymbol{\pi}^j_{b+1}, t_{b+1})$ is known for all values $j = 1, 2, ..., G^{k-1}$ on the discretization grid. Monte Carlo approximation of the expectation in the objective function

$$(1-\psi_b)^{1-\gamma} + \beta\psi_b^{1-\gamma} E_{t_b}\left[\left((1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right]$$

requires drawing N random samples of asset returns $\{\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^j_b)\}_{n=1}^N$ from the $(b+1)\varphi$-step-ahead joint density conditional on period-t parameter estimates, $\widehat{\boldsymbol{\theta}}_t = \{\hat{\boldsymbol{\mu}}_t, \widehat{\boldsymbol{\Omega}}_t, \hat{\mathbf{P}}_t\}$ assuming that, at each point, $\boldsymbol{\pi}^j_{b,n}$ is optimally updated to $\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}^j_{b,n})$. The algorithm consists of the following steps:

1. For each possible value of the current regime, s_b, simulate N $\varphi-$period returns in calendar time $\{\mathbf{R}_{b+1,n}(s_b) \equiv \sum_{j=1}^{\varphi} \mathbf{r}_{t_b+j,n}(s_b)\}_{n=1}^N$ from the regime switching model

$$\mathbf{r}_{t_b+j,n}(s_b) = \hat{\boldsymbol{\mu}}_{s_{t_b+j}} + \boldsymbol{\varepsilon}_{t_b+j,n}, \quad \boldsymbol{\varepsilon}_{t_b+j,n} \sim N(0, \hat{\boldsymbol{\Omega}}_{s_{t_b+j}}).$$

   At all rebalancing points this simulation allows for regime switching. For example, if we start in regime 1, between t_b and t_b+1 there is a chance $\hat{p}_{12} \equiv \mathbf{e}_1'\hat{\mathbf{P}}\mathbf{e}_2$ of switching to regime 2, and a chance $\hat{p}_{11} \equiv \mathbf{e}_1'\hat{\mathbf{P}}\mathbf{e}_1$ of staying in regime 1. At each point in time $\hat{\mathbf{P}}_t$ governs possible switches.

2. Combine the simulated $\varphi-$period returns $\{\mathbf{R}_{b+1,n}\}_{n=1}^N$ in a random sample of size N, using the probability weights contained in the row vector $\boldsymbol{\pi}^j_{b,n}$

$$\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_{b,n}) = \sum_{j=1}^{k}(\mathbf{e}_j'\boldsymbol{\pi}^i_{b,n})\mathbf{R}_{b+1,n}(S_b = j).$$

3. Update the future regime probabilities perceived by the investor using the formula

$$\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}^i_{b,n}) = \frac{((\boldsymbol{\pi}^i_{b,n})'\hat{\mathbf{P}}^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_{b,n}); \hat{\boldsymbol{\theta}}_t)}{[((\boldsymbol{\pi}^i_{b,n})'\hat{\mathbf{P}}^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_{b,n}); \hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k},$$

   where $\hat{\mathbf{P}}^{\varphi} \equiv \prod_{i=1}^{\varphi} \hat{\mathbf{P}}$. This gives an $N \times k$ matrix $\{\boldsymbol{\pi}'_{b+1,n}(\boldsymbol{\pi}^i_{b,n})\}_{n=1}^N$, whose rows correspond to simulated vectors of perceived regime probabilities at time t_{b+1}.

4. For all $n = 1, 2, ..., N$, calculate the value $\tilde{\boldsymbol{\pi}}^i_{b+1,n}$ on the discretization grid ($i = 1,\ 2, ...,\ G^{k-1}$) closest to $\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}^i_{b,n})$ using the distance measure $\sum_{j=1}^{k-1} |\mathbf{e}_j'\boldsymbol{\pi}^i_{b+1} - \mathbf{e}_j'\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}^i_{b,n})|$, i.e.

$$\tilde{\boldsymbol{\pi}}^i_{b+1,n}(\boldsymbol{\pi}^i_{b,n}) \equiv \underset{\mathbf{x}\in\times_{j=1}^{k-1}[0,1]}{\arg\min} \sum_{j=1}^{k-1} |\mathbf{e}_j'\mathbf{x} - \mathbf{e}_j'\boldsymbol{\pi}_{b+1,n}|.$$

Knowing the vector $\{\tilde{\boldsymbol{\pi}}^i_{b+1,n}(\boldsymbol{\pi}^i_{b,n})\}_{n=1}^N$ we can build $\{Q(\boldsymbol{\pi}^{(i,n)}_{b+1}, t_{b+1})\}_{n=1}^N$, where $\boldsymbol{\pi}^{(i,n)}_{b+1} \equiv \tilde{\boldsymbol{\pi}}^i_{b+1,n}(\boldsymbol{\pi}^i_{b,n})$ is a function of the initial vector of regime probabilities $\boldsymbol{\pi}^i_{b,n}$ on the simulated path.[23]

5. Solve the program

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}^j_b),\psi_b(\boldsymbol{\pi}^j_b)} (1-\psi_b)^{1-\gamma} + \beta\psi_b^{1-\gamma} N^{-1} \sum_{n=1}^N \left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \times \right.\right.$$
$$\left.\left. \times \exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^j_{b,n}) + \varphi r^f \boldsymbol{\iota}_3\right)\right\}^{1-\gamma} Q(\boldsymbol{\pi}^{(j,n)}_{b+1}, t_{b+1})\right],$$

which for large values of N provides an arbitrarily precise Monte Carlo approximation to expected utility. The value function evaluated at the optimal portfolio weights $\boldsymbol{\omega}_b(\boldsymbol{\pi}^j_b)$ and savings ratio $\psi_b(\boldsymbol{\pi}^j_b)$ defines $Q(\boldsymbol{\pi}^j_b, t_b)$ at the jth point on the initial grid.

The algorithm is applied to all possible values $\boldsymbol{\pi}^j_b$ on the discretization grid until all values of $Q(\boldsymbol{\pi}^j_b, t_b)$ are obtained for $j = 1,\ 2, ...,\ G^{k-1}$. It is then iterated backwards until $t_{b+1} = t + \varphi$. At this stage the algorithm is applied one last time, taking $Q(\boldsymbol{\pi}^j_{t+\varphi}, t+\varphi)$ as given and using the actual row vector of smoothed regime probabilities $\boldsymbol{\pi}_t$. The resulting $\boldsymbol{\omega}_t$, ψ_t are the desired optimal portfolio allocation and the optimal savings rate, respectively, while $Q(\boldsymbol{\pi}_t, t)$ is the optimal value function.

In the buy-and-hold case ($\varphi = T - t$) step 2 is replaced with a simulation routine that for each possible future regime, s_b, simulates N asset returns of length T, $\{\mathbf{R}_{T,n}(s_b) \equiv \sum_{i=1}^T \mathbf{r}_{t+i,n}(s_b)\}_{n=1}^N$ from the Markov switching model

$$\mathbf{r}_{t+i,n}(s_b) = \hat{\boldsymbol{\mu}}_{s_{t+i}} + \boldsymbol{\varepsilon}_{t+i,n}, \quad \boldsymbol{\varepsilon}_{t+i,n} \sim N(0, \hat{\boldsymbol{\Omega}}_{s_{t+i}}).$$

In other words, a matrix of monthly returns $\left\{\{\mathbf{r}_{t+i,n}(s_b)\}_{n=1}^N\right\}_{i=1}^T$ is first drawn and then summed into N long-term asset returns $\{\mathbf{R}_{T,n}(s_b)\}_{n=1}^N$. Steps 1 and 4-6 are irrelevant in the buy-and-hold case since the objective simplifies to

$$\max_{\boldsymbol{\omega}_t,\psi_t} \frac{(1-\psi_t)^{1-\gamma}}{1-\gamma} + \beta\psi_t^{1-\gamma} N^{-1} \sum_{n=1}^N \left\{ \frac{\left[(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_3)\exp\left(Tr^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{T,n}(\boldsymbol{\pi}^j_b) + \varphi r^f \boldsymbol{\iota}_3\right)\right]^{1-\gamma}}{1-\gamma} \right\},$$

where $\mathbf{R}_{T,n} = \sum_{i=1}^4 (\boldsymbol{\pi}_t'\mathbf{e}_i)\mathbf{R}_{T,n}(s_b = i)$. This makes computations much faster in this case.

[23] This step may be avoided when $Q(\boldsymbol{\pi}^i_{b+1}, t_{b+1})$ is constant for all values on the discretization grid. This happens when $t_{b+1} = T$ and implies that the portfolio weights determined at step $b+1$ $\{\boldsymbol{\omega}_{b+1}(\boldsymbol{\pi}^i_{b+1})\}$ are invariant to changes in $\boldsymbol{\pi}^i_{b+1}$. In this case the step simplifies to

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}^i_b)} N^{-1} \sum_{n=1}^N \left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_b) + \varphi r^f \mathbf{e}_1\right)\right\}^{1-\gamma}\right].$$

Finally, extending these methods to the case with additional predictor variables is straightforward and only implies generalizing step 1 to simulate on a suitable grid N $\varphi-$period returns $\{\mathbf{R}_{b+1,n}(s_b)\}_{n=1}^{N}$ for each possible value of the regime s_b and each possible configuration of the state vector $(\mathbf{r}_b' \ \mathbf{z}_b')'$:

$$\begin{pmatrix} \mathbf{r}_{t_b+j,n}(s_b) \\ \mathbf{z}_{t_b+j,n}(s_b) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\mu}}_{s_{t_b+j}} \\ \hat{\boldsymbol{\mu}}_{zs_{t_b+j}} \end{pmatrix} + \hat{\mathbf{A}}_{s_t} \begin{pmatrix} \mathbf{r}_{t_b+j-1,n}(s_b) \\ \mathbf{z}_{t_b+j-1,n}(s_b) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_{t_b+j,n} \\ \boldsymbol{\varepsilon}_{zt_b+j,n} \end{pmatrix}$$

where $\mathbf{R}_{b+1,n}(s_b) \equiv \sum_{j=1}^{\varphi} \mathbf{r}_{t_b+j,n}(s_b)$ and $(\boldsymbol{\varepsilon}_{t_b+j,n}' \ \boldsymbol{\varepsilon}_{zt_b+j,n}')' \sim N(0, \hat{\boldsymbol{\Omega}}_{s_{t_b+j,n}})$. Step 3 is virtually identical, the only difference being that the updating of $\boldsymbol{\pi}_b^i$ must now incorporate a likelihood function $\boldsymbol{\eta}(\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i); \hat{\boldsymbol{\theta}}_t)$ defined over $\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i) \equiv (\mathbf{r}_{b+1,n}'(\boldsymbol{\pi}_b^i) \ \mathbf{z}_{b+1,n}'(\boldsymbol{\pi}_b^i))'$. Step 4 must be adjusted to define distances on the discretization grid to also account for values of $\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i)$ generated on each of the simulated paths.

Appendix B - Discretization and Monte Carlo Methods

This appendix addresses some of the issues arising from application of discretization and Monte Carlo methods to multivariate Gaussian mixtures and investigates our choice of the number of grid points, G, as well as the number of Monte Carlo simulations, N, used to approximate the integrals involved in the computation of expected utility.

B*1. Myopic Savings and Portfolio Choices*

When $\gamma = 1$ and no short-sale constraints are imposed, (21)-(22) admit a closed-form solution for savings decisions and portfolio choices.[24] Under log-utility the value function can be written as

$$J(W_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j, \psi_j\}_{j=b}^{B-1}} E_{t_b}\left[\sum_{i=0}^{B} \beta^j \left(\ln(1-\psi_{b+i}) + \ln W_{b+i}\right)\right],$$

while the Bellman equation and the dynamic budget constraint are

$$\begin{aligned} J(W_b, \boldsymbol{\pi}_b, t_b) &\equiv \max_{\boldsymbol{\omega}_b, \psi_b} \beta^b \ln(1-\psi_b) + \beta^b \ln W_b + \beta^{b+1} E_{t_b}\left[J(W_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1})\right], \qquad (26) \\ W_{b+1} &= W_b \psi_b \left[(1-\boldsymbol{\omega}_b' \iota_3) \exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right]. \end{aligned}$$

Under logarithmic utility the optimal savings ratio is available in closed-form,

$$\psi_b = \frac{\beta - \beta^{B-b+1}}{1-\beta^{B-b+1}}. \qquad (27)$$

In each period a deterministic fraction $(\beta - \beta^{B-b+1})/(1-\beta^{B-b+1})$ is therefore saved. The first order conditions with respect to the optimal portfolio weights are

$$\begin{aligned} E_{t_b}\left[\frac{\psi W_b}{W_{b+1}}\left(\exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right) - \boldsymbol{\iota}_3 \exp\left(\varphi r^f\right)\right)\right] &= \mathbf{0}, \\ E_{t_b}\left[\frac{\exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right) - \boldsymbol{\iota}_3 \exp\left(\varphi r^f\right)}{\left[(1-\boldsymbol{\omega}'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right]}\right] &= \mathbf{0}. \end{aligned}$$

The investor behaves myopically and ignores time-variation in future beliefs as well as time-variations in the joint distribution of asset returns.

[24]For concreteness we focus on the case without autoregressive terms or predictor variables ($\mathbf{A}_s = \mathbf{O}$ for all s, $\mathbf{z}_t = 1$).

The numerical procedure outlined in Appendix A does not utilize the simplifications resulting from logarithmic utility. The same backward induction techniques are applied independently of the specific value assumed by γ. Assuming that the problem has already been solved backwards at the rebalancing times t_{B-1}, ..., t_{b+1}, step 5 in Appendix A iterates over $\boldsymbol{\pi}_b^j$ on the grid ($j = 1, 2, ..., G^{k-1}$) providing a solution to the program

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^j), \psi_b(\boldsymbol{\pi}_b^j)} \beta \ln \psi_b N^{-1} \sum_{n=1}^{N} \ln \left[(1 - \boldsymbol{\omega}_b' \iota_3) \exp\left(\varphi r^f\right) + \exp\left(\boldsymbol{\omega}_b'(\varphi r^f + \mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j))\right) \right] \times \\ \times Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1}) + \ln(1 - \psi_b).$$

To evaluate the precision of our numerical procedure we calculate for each of the four regimes optimal savings and asset allocation choices for investment horizons of 1 and 60 months using the Monte Carlo approach and assuming annual rebalancing ($\varphi = 12$). We vary the following parameters:

1. The number of simulations N grows from $1,000$ to $50,000$ in steps of 2,000.

2. The number of points G on the discretization grid is varied in the range $G = 2, 4, 5, 8, 10, 20$.

We compare the optimal savings choices obtained from equation (27) to the optimal decisions calculated numerically. Apart from the extreme case where $G = 2$, the number of grid points used in the solution to the dynamic program does not seem to be crucial: Provided that the number of simulations exceeds $N = 20,000$, ψ_t becomes very similar to the value in (27). Results are also not overly sensitive to the investment horizon T. We conclude that $G \geq 5$ and $N \geq 20,000$ guarantee sufficient accuracy in the calculations of optimal consumption choices. In the paper we use $G = 5$ and set N at $30,000$ or $50,000$.

B*2. Sampling Errors in the Buy-and-Hold Portfolio Optimization.*

The second experiment is similar in spirit to Barberis (2000, pp. 262-263): For each of the four regimes, we calculate optimal asset allocation choices for investment horizons of 6 and 60 months using our Monte Carlo approach as the number of simulations N grows from $1,000$ to $50,000$ in steps of 2,000. For each value of N, we repeat the experiment 10 times.

The first row of plots in Figure B1 shows the overall allocation to stocks as a function of the simulation trial (indexed by an integer between 1 and 10) for $T = 6$ and 60 months. For $N \leq 20,000$ sampling errors dominate in all regimes and random variation in the optimal allocation is substantial. When $20,000 \leq N \leq 30,000$ the quality of the approximation depends on the current state probabilities. In particular, in the longer-lived regimes (2 and 3) sampling errors may still be rather large, on the order of 3-4%. Only when $N \geq 30,000$ do the Monte Carlo simulation errors become sufficiently small in the more persistent regimes. The bottom plots in Figure B1 extend the exercise to small stocks and long-term bonds. $N = 30,000$ guarantees a substantial reduction in the incidence of sampling error, while a further increase to $N = 40,000$ or $50,000$ only has second-order effects.

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Table 1

Specification Tests for Regime Switching Models

The table reports tests for the transformed z-scores generated by the multivariate regime-switching model:

$$y_t = \boldsymbol{\mu}^*_{s_t} + \sum_{j=1}^{p} A^*_{js_t} y_{t-j} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{y}_t$ collects returns on a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills, as well as exogenous predictors (dividend yield), $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients at lag $j \geq 1$ in state S_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Omega^*_{s_t})$.The unobserved state S_t is governed by a first-order Markov chain that can assume *k* distinct values. The sample period is 1954:01 – 1999:12. The tests are based on the principle that under the null of correct specification of the model, the probability integral transform of the standardized forecast errors should follow an IID uniform distribution over the interval (0,1). A further Gaussian transform described in Berkowitz (2001) is applied to perform LR tests of the null that the transformed z-scores are IIN(0,1) distributed. In the table, MSIAH(*k*,*p*) denotes a *k*-state multivariate regime switching (MS), with shifts in intercepts (I) and covariance matrices (H), and *p* autoregressive (A) lags. Skew and Kurt report the skewness and kurtosis of the z-scores. Jarque-Bera values provide a test of normality while the likelihood ratio (LR) values provide tests for zero correlation in levels and squares of the standardized residuals.

Model	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6
	Panel A – Return Model																	
	Large Caps Portfolio						**Small Caps Portfolio**						**Long-Term Bonds**					
MSIA(1,0)	0.051	7.593	**473.65** **(0.000)**	1.846 (0.397)	**24.222** **(0.000)**	**30.024** **(0.000)**	-0.385	4.978	**101.72** **(0.000)**	1.928 (0.381)	4.076 (0.253)	10.044 (0.123)	0.386	4.671	**76.566** **(0.000)**	2.278 (0.320)	**7.824** **(0.050)**	**17.004** **(0.009)**
MSIA(1,1)	0.271	8.182	**601.73** **(0.000)**	0.000 (1.000)	1.994 (0.574)	8.718 (0.190)	-0.355	4.665	**72.66** **(0.000)**	0.000 (1.000)	1.916 (0.590)	6.964 (0.324)	0.334	4.402	**53.440** **(0.000)**	0.000 (1.000)	2.496 (0.476)	11.374 (0.077)
MSIA (2,0)	0.023	3.599	**8.018** **(0.018)**	0.002 (0.999)	1.952 (0.582)	6.642 (0.355)	0.600	6.690	**336.93** **(0.000)**	0.132 (0.936)	**23.010** **(0.000)**	**29.124** **(0.000)**	0.334	4.528	**62.233** **(0.000)**	0.012 (0.994)	5.538 (0.136)	**14.678** **(0.023)**
MSIH (2,0)	-0.190	3.507	**8.891** **(0.012)**	0.078 (0.962)	2.118 (0.548)	6.930 (0.327)	0.050	7.606	**469.64** **(0.000)**	**37.344** **(0.000)**	**19.428** **(0.000)**	**19.214** **(0.004)**	0.077	3.166	1.132 (0.568)	0.038 (0.981)	7.400 (0.060)	**13.452** **(0.037)**
MSIAH (2,1)	-0.160	3.463	**6.752** **(0.034)**	0.042 (0.979)	2.578 (0.461)	6.888 (0.331)	0.043	4.197	**30.018** **(0.000)**	0.084 (0.959)	2.020 (0.568)	5.496 (0.482)	0.097	3.209	1.730 (0.421)	0.054 (0.973)	2.026 (0.567)	9.656 (0.140)
MSIA (4,0)	-0.169	4.374	**43.459** **(0.000)**	0.126 (0.939)	2.430 (0.488)	7.366 (0.289)	-0.107	5.597	**147.37** **(0.000)**	0.114 (0.945)	**14.064** **(0.003)**	**21.656** **(0.001)**	-0.163	3.894	**19.662** **(0.000)**	0.002 (0.999)	3.498 (0.321)	11.876 (0.065)
MSIH (4,0)	0.076	3.484	5.397 (0.067)	0.122 (0.941)	3.090 (0.378)	7.378 (0.287)	0.086	3.987	**21.028** **(0.000)**	0.030 (0.985)	**12.254** **(0.007)**	**15.622** **(0.016)**	0.028	3.049	0.119 (0.942)	0.042 (0.979)	5.694 (0.127)	9.582 (0.143)
MSIAH (4,1)	0.152	3.922	**18.239** **(0.000)**	**6.154** **(0.046)**	**8.062** **(0.045)**	12.154 (0.057)	0.888	11.50	**1460.9** **(0.000)**	**9.444** **(0.009)**	**32.166** **(0.000)**	**35.822** **(0.000)**	0.018	2.739	1.343 (0.511)	2.890 (0.236)	**13.994** **(0.003)**	**18.738** **(0.005)**
MSIH (6,0)	0.016	3.269	1.434 (0.488)	0.018 (0.991)	3.424 (0.331)	8.146 (0.228)	0.053	3.558	**6.283** **(0.043)**	0.302 (0.860)	7.286 (0.063)	**14.232** **(0.027)**	0.016	3.245	1.183 (0.553)	0.142 (0.932)	3.856 (0.277)	9.368 (0.154)

Table 1 (continued)

Specification Tests for Regime Switching Models

Model	Skew	Kurt.	Jarque -Bera	LR$_2$	LR$_3$	LR$_6$	Skew	Kurt.	Jarque -Bera	LR$_2$	LR$_3$	LR$_6$	Skew	Kurt.	Jarque -Bera	LR$_2$	LR$_3$	LR$_6$
	Panel B – Return and Dividend Yield Model																	
	Large Caps Portfolio						**Small Caps Portfolio**						**Long-Term Bonds**					
MSIA(1,0)	0.051	7.593	**473.65 (0.000)**	1.846 (0.397)	**24.22 (0.000)**	**30.02 (0.000)**	-0.385	4.978	**101.72 (0.000)**	1.928 (0.381)	4.076 (0.253)	10.044 (0.123)	0.386	4.671	**76.57 (0.000)**	2.278 (0.320)	**7.824 (0.050)**	**17.004 (0.009)**
MSIA(1,1)	-0.287	4.864	**87.32 (0.000)**	**31.56 (0.000)**	**32.63 (0.000)**	**34.05 (0.190)**	0.305	8.294	**630.60 (0.000)**	**7.848 (0.020)**	**8.742 (0.033)**	9.136 (0.166)	0.324	4.395	**52.60 (0.000)**	**6.345 (0.042)**	7.727 (0.052)	9.715 (0.137)
MSIA (2,0)	-0.385	4.976	**99.50 (0.000)**	0.002 (0.999)	2.142 (0.543)	8.116 (0.230)	0.043	7.548	**457.80 (0.000)**	0.003 (0.987)	**20.91 (0.000)**	**26.814 (0.000)**	0.389	4.674	**75.40 (0.000)**	0.000 (1.000)	5.568 (0.135)	**14.830 (0.022)**
MSIA (2,1)	-0.229	5.247	**109.09 (0.000)**	0.022 (0.989)	2.274 (0.518)	6.862 (0.334)	0.196	8.152	**553.99 (0.000)**	**8.434 (0.015)**	**11.598 (0.009)**	**17.760 (0.007)**	0.259	4.283	**39.70 (0.000)**	**16.34 (0.000)**	**17.454 (0.000)**	**23.344 (0.001)**
MSIAH (2,0)	-0.497	5.084	**115.76 (0.000)**	0.058 (0.971)	2.254 (0.521)	8.382 (0.211)	-0.124	7.065	**360.01 (0.000)**	0.062 (0.969)	**23.76 (0.000)**	**30.330 (0.000)**	0.245	3.890	**22.42 (0.000)**	0.042 (0.979)	5.540 (0.136)	10.338 (0.111)
MSIAH (2,1)	-0.098	4.797	**66.49 (0.000)**	0.090 (0.956)	2.280 (0.516)	6.684 (0.351)	0.092	8.512	**618.50 (0.000)**	0.054 (0.973)	1.908 (0.592)	6.988 (0.322)	0.189	4.190	**31.73 (0.000)**	0.052 (0.974)	1.900 (0.593)	8.258 (0.220)
MSIAH (3,1)	-0.083	4.029	**20.57 (0.000)**	0.048 (0.976)	2.452 (0.484)	5.556 (0.475)	-0.022	5.855	**154.32 (0.000)**	0.020 (0.990)	2.030 (0.566)	6.580 (0.361)	0.026	3.630	**7.56 (0.023)**	0.068 (0.967)	2.002 (0.575)	7.970 (0.240)
MSIAH (3,2)	-0.115	3.802	**11.77 (0.003)**	0.072 (0.965)	3.006 (0.391)	5.362 (0.498)	0.034	5.101	**74.580 (0.000)**	0.060 (0.970)	2.206 (0.531)	7.241 (0.302)	-0.008	3.559	5.27 (0.072)	0.082 (0.960)	2.234 (0.525)	7.126 (0.309)
MSIA (4,1)	0.172	3.445	5.897 (0.052)	0.030 (0.985)	2.158 (0.540)	5.738 (0.453)	0.557	6.912	**308.96 (0.000)**	0.058 (0.971)	2.566 (0.463)	5.804 (0.446)	0.177	4.085	**24.35 (0.000)**	**16.73 (0.000)**	**17.222 (0.001)**	**23.306 (0.001)**
MSIAH (4,1)	-0.106	3.423	3.908 (0.142)	0.168 (0.919)	2.212 (0.530)	5.266 (0.510)	0.116	3.468	4.763 (0.092)	0.014 (0.993)	2.192 (0.534)	5.898 (0.435)	0.045	3.505	4.59 (0.101)	0.008 (0.996)	1.850 (0.604)	6.816 (0.338)
MSIAH (4,2)	0.010	3.194	0.558 (0.757)	0.052 (0.974)	3.550 (0.314)	6.046 (0.418)	0.161	3.748	**9.770 (0.008)**	0.312 (0.856)	2.792 (0.425)	5.592 (0.470)	0.127	3.194	1.50 (0.472)	0.818 (0.664)	3.620 (0.306)	7.882 (0.251)

Table 2

Estimates of Regime Switching Model for Stock and Bond Returns

This table reports the estimation output for the regime switching model:

$$\mathrm{r}_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0},\Omega_{s_t})$ is the vector of return innovations. The unobserved state variable S_t is governed by a first-order Markov chain that can assume k values. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The sample is 1954:01 – 1999:12. Panel A refers to the case (k = 1) and represents a single-state benchmark. The data reported on the diagonals of the correlation matrices are annualized volatilities. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – Single State Model			
	Large caps	Small caps	Long-term bonds	
1. Mean excess return	0.0066 (0.0018)	0.0082 (0.0026)	0.0008 (0.0009)	
2. Correlations/Volatilities				
Large caps	0.1428***			
Small caps	0.7215**	0.2129***		
Long-term bonds	0.2516	0.1196	0.0748***	
	Panel B – Four State Model			
	Large caps	Small caps	Long-term bonds	
1. Mean excess return				
Regime 1 (crash)	-0.0510 (0.0146)	-0.0410 (0.0219)	-0.0131 (0.0047)	
Regime 2 (slow growth)	0.0069 (0.0027)	0.0008 (0.0033)	0.0009 (0.0016)	
Regime 3 (bull)	0.0116 (0.0032)	0.0187 (0.0048)	-0.0023 (0.0007)	
Regime 4 (recovery)	0.0519 (0.0055)	0.0478 (0.0098)	0.0136 (0.0033)	
2. Correlations/Volatilities				
Regime 1 (crash):				
Large caps	0.1625***			
Small caps	0.8233***	0.2479***		
Long-term bonds	-0.4060*	-0.2590	0.0902***	
Regime 2 (slow growth):				
Large caps	0.1118***			
Small caps	0.7655***	0.1099***		
Long-term bonds	0.2043***	0.1223	0.0688***	
Regime 3 (bull):				
Large caps	0.1133***			
Small caps	0.6707***	0.1730***		
Long-term bonds	0.1521	-0.0976	0.0261***	
Regime 4 (recovery):				
Large caps	0.1479***			
Small caps	0.5013***	0.2429***		
Long-term bonds	0.3692***	-0.0011	0.1000***	
3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (crash)	0.4940 (0.1078)	0.0001 (0.0001)	0.02409 (0.0417)	0.4818
Regime 2 (slow growth)	0.0483 (0.0233)	0.8529 (0.0403)	0.0307 (0.0110)	0.0682
Regime 3 (bull)	0.0439 (0.0252)	0.0701 (0.0296)	0.8822 (0.0403)	0.0038
Regime 4 (recovery)	0.0616 (0.0501)	0.1722 (0.0718)	0.0827 (0.0498)	0.6836

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 3

Effects of Rebalancing on Asset Allocation

This table reports the optimal weight on stocks (small and large) and bonds as a function of the rebalancing frequency φ for an investor with power utility and a constant relative risk aversion coefficient of 5. Excess returns are assumed to be generated by the regime switching model

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations.

Rebalancing Frequency φ	Investment Horizon T (in months)					
	A - Optimal Allocation to Stocks					
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
	Crash regime 1					
φ = T (buy-and-hold)	0.00	0.24	0.34	0.48	0.58	0.60
φ = 24 months	---	---	---	---	0.50	0.50
φ = 12 months	---	---	---	0.37	0.39	0.40
φ = 6 months	---	---	0.28	0.31	0.33	0.34
φ = 3 months	---	0.00	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
	Slow growth regime 2					
φ = T (buy-and-hold)	1.00	0.68	0.65	0.65	0.65	0.64
φ = 24 months	---	---	---	---	0.70	0.80
φ = 12 months	---	---	---	0.72	0.82	0.93
φ = 6 months	---	---	0.71	0.77	0.88	0.96
φ = 3 months	---	0.92	0.85	0.89	0.95	0.99
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
	Bull regime 3					
φ = T (buy-and-hold)	1.00	0.67	0.66	0.65	0.65	0.65
φ = 24 months	---	---	---	---	0.72	0.83
φ = 12 months	---	---	---	0.74	0.85	0.88
φ = 6 months	---	---	0.74	0.80	0.90	0.95
φ = 3 months	---	0.94	0.96	0.98	1.00	1.00
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
	Recovery regime 4					
φ = T (buy-and-hold)	1.00	0.82	0.71	0.69	0.68	0.66
φ = 24 months	---	---	---	---	0.71	0.74
φ = 12 months	---	---	---	0.72	0.74	0.77
φ = 6 months	---	---	0.75	0.79	0.82	0.85
φ = 3 months	---	0.98	1.00	1.00	1.00	1.00
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
	Steady-state probabilities					
φ = T (buy-and-hold)	1.00	0.73	0.68	0.67	0.65	0.64
φ = 24 months	---	---	---	---	0.71	0.77
φ = 12 months	---	---	---	0.73	0.78	0.81
φ = 6 months	---	---	0.78	0.81	0.84	0.83
φ = 3 months	---	0.88	0.85	0.84	0.84	0.84
φ = 1 month	1.00	0.98	0.98	0.98	0.98	0.98

Table 3 (continued)

Effects of Rebalancing

Rebalancing Frequency φ	Investment Horizon T (in months)					
B - Optimal Allocation to Long-Term Bonds						
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
Crash regime 1						
φ = T (buy-and-hold)	0.00	0.34	0.29	0.19	0.12	0.08
φ = 24 months	---	---	---	---	0.16	0.10
φ = 12 months	---	---	---	0.21	0.17	0.11
φ = 6 months	---	---	0.28	0.18	0.15	0.10
φ = 3 months	---	0.18	0.16	0.13	0.11	0.05
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Slow growth regime 2						
φ = T (buy-and-hold)	0.00	0.32	0.34	0.19	0.14	0.08
φ = 24 months	---	---	---	---	0.17	0.13
φ = 12 months	---	---	---	0.20	0.14	0.01
φ = 6 months	---	---	0.21	0.13	0.04	0.00
φ = 3 months	---	0.05	0.13	0.04	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Bull regime 3						
φ = T (buy-and-hold)	0.00	0.00	0.00	0.00	0.00	0.00
φ = 24 months	---	---	---	---	0.05	0.00
φ = 12 months	---	---	---	0.06	0.03	0.00
φ = 6 months	---	---	0.07	0.02	0.00	0.00
φ = 3 months	---	0.02	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Recovery regime 4						
φ = T (buy-and-hold)	0.00	0.17	0.12	0.10	0.08	0.08
φ = 24 months	---	---	---	---	0.01	0.01
φ = 12 months	---	---	---	0.00	0.00	0.00
φ = 6 months	---	---	0.00	0.00	0.00	0.00
φ = 3 months	---	0.00	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Steady-state probabilities						
φ = T (buy-and-hold)	0.00	0.03	0.04	0.05	0.07	0.06
φ = 24 months	---	---	---	---	0.10	0.12
φ = 12 months	---	---	---	0.08	0.12	0.14
φ = 6 months	---	---	0.07	0.10	0.16	0.17
φ = 3 months	---	0.06	0.15	0.12	0.11	0.10
φ = 1 month	0.00	0.02	0.02	0.02	0.02	0.02

Table 4

Optimal Consumption-Wealth Ratio – Effects of Rebalancing

This table reports the optimal consumption-wealth ratio as a function of the rebalancing frequency φ and the investment horizon T for an investor with power utility, constant relative risk aversion coefficient of 5, and (annualized) subjective rate of time preference of 5%. Excess returns are assumed to be generated by the regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t ,$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations.

Rebalancing Frequency φ	**Investment Horizon T (in months)**					
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
Crash regime 1						
φ = T (buy-and-hold)	0.50	0.51	0.52	0.54	0.60	0.69
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.52	0.34	0.18	0.10
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.50	0.32	0.19	0.11	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Slow growth regime 2						
φ = T (buy-and-hold)	0.51	0.51	0.52	0.54	0.60	0.69
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.51	0.51	0.52	0.46	0.28	0.21
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.51	0.35	0.21	0.13	0.03	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Bull regime 3						
φ = T (buy-and-hold)	0.50	0.51	0.54	0.56	0.61	0.70
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.54	0.36	0.20	0.14
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.50	0.33	0.20	0.11	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Recovery regime 4						
φ = T (buy-and-hold)	0.50	0.51	0.52	0.54	0.60	0.67
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.52	0.38	0.22	0.14
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 month	0.50	0.33	0.20	0.12	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03

Table 5

Effect of Parameter Estimation Uncertainty on Asset Allocation

This table reports confidence bands for a buy-and-hold investor's optimal portfolio weights at different investment horizons, T, assuming a constant relative risk aversion coefficient of 5. Under regime switching, portfolio returns are assumed to be generated by the model

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations. In the IID case, *k* = 1. Boldfaced blocks of cells indicate a portfolio weight confidence interval that fails to include the IID weight.

		Investment Horizon T						
		T=1	T=6	T=24	T=48	T=72	T=96	T=120
A: Allocation to Small Stocks								
Crash regime 1	Mean + 1*SD	0.000	**0.319**	**0.393**	**0.392**	**0.395**	**0.390**	**0.394**
	Mean	0.000	**0.173**	**0.230**	**0.228**	**0.228**	**0.225**	**0.226**
	Mean - 1*SD	0.000	**0.028**	**0.067**	**0.063**	**0.061**	**0.060**	**0.058**
Slow growth regime 2	Mean + 1*SD	0.211	0.277	**0.357**	**0.375**	**0.385**	**0.383**	**0.383**
	Mean	0.061	0.127	**0.197**	**0.212**	**0.217**	**0.218**	**0.217**
	Mean - 1*SD	0.000	0.000	**0.037**	**0.049**	**0.050**	**0.053**	**0.052**
Bull regime 3	Mean + 1*SD	**0.915**	**0.530**	**0.432**	**0.410**	**0.404**	**0.403**	**0.401**
	Mean	**0.632**	**0.313**	**0.258**	**0.242**	**0.235**	**0.233**	**0.231**
	Mean - 1*SD	**0.349**	**0.096**	**0.083**	**0.073**	**0.067**	**0.064**	**0.060**
Recovery regime 4	Mean + 1*SD	**1.000**	**0.607**	**0.457**	**0.424**	**0.417**	**0.410**	**0.411**
	Mean	**0.890**	**0.406**	**0.279**	**0.252**	**0.245**	**0.238**	**0.236**
	Mean - 1*SD	**0.706**	**0.205**	**0.101**	**0.080**	**0.073**	**0.066**	**0.061**
Steady-state	Mean + 1*SD	**1.000**	**0.573**	**0.447**	**0.418**	**0.407**	**0.405**	**0.401**
	Mean	**0.827**	**0.361**	**0.270**	**0.247**	**0.238**	**0.235**	**0.231**
	Mean - 1*SD	**0.634**	**0.149**	**0.092**	**0.076**	**0.069**	**0.065**	**0.061**
B: Allocation to Large Stocks								
Crash regime 1	Mean + 1*SD	**0.050**	**0.290**	0.497	0.553	0.573	0.579	0.590
	Mean	**0.005**	**0.114**	0.275	0.323	0.341	0.347	0.355
	Mean - 1*SD	**0.000**	**0.000**	0.053	0.093	0.109	0.116	0.119
Slow growth regime 2	Mean + 1*SD	**1.000**	0.709	0.629	0.616	0.613	0.611	0.613
	Mean	**0.834**	0.470	0.395	0.384	0.380	0.379	0.380
	Mean - 1*SD	**0.621**	0.232	0.161	0.151	0.148	0.147	0.147
Bull regime 3	Mean + 1*SD	0.630	0.703	0.632	0.620	0.616	0.619	0.616
	Mean	0.351	0.441	0.393	0.384	0.382	0.384	0.381
	Mean - 1*SD	0.073	0.179	0.154	0.148	0.147	0.148	0.146
Recovery regime 4	Mean + 1*SD	**0.275**	0.500	0.570	0.591	0.592	0.603	0.604
	Mean	**0.101**	0.268	0.336	0.356	0.360	0.368	0.369
	Mean - 1*SD	**0.000**	0.039	0.102	0.122	0.128	0.132	0.135
Steady-state	Mean + 1*SD	0.724	0.648	0.611	0.608	0.609	0.610	0.608
	Mean	0.174	0.406	0.386	0.381	0.380	0.378	0.380
	Mean - 1*SD	0.195	0.145	0.135	0.137	0.139	0.139	0.140

Table 5 - continued

		Investment Horizon T						
		T=1	T=6	T=24	T=48	T=72	T=96	T=120
C: Allocation to Bonds								
Crash regime 1	Mean + 1*SD	**0.033**	**0.481**	**0.406**	**0.375**	**0.363**	**0.360**	**0.356**
	Mean	**0.000**	**0.264**	**0.221**	**0.200**	**0.190**	**0.190**	**0.186**
	Mean - 1*SD	**0.000**	**0.047**	**0.036**	**0.024**	**0.018**	**0.019**	**0.015**
Slow growth regime 2	Mean + 1*SD	**0.229**	**0.383**	**0.359**	**0.348**	**0.345**	**0.343**	**0.343**
	Mean	**0.084**	**0.206**	**0.191**	**0.183**	**0.180**	**0.179**	**0.178**
	Mean - 1*SD	**0.000**	**0.028**	**0.025**	**0.019**	**0.015**	**0.014**	**0.012**
Bull regime 3	Mean + 1*SD	**0.000**	**0.043**	**0.221**	**0.276**	**0.296**	**0.307**	**0.313**
	Mean	**0.000**	**0.010**	**0.095**	**0.130**	**0.143**	**0.151**	**0.156**
	Mean - 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
Recovery regime 4	Mean + 1*SD	**0.037**	**0.401**	**0.371**	**0.357**	**0.350**	**0.346**	**0.347**
	Mean	**0.006**	**0.230**	**0.203**	**0.191**	**0.185**	**0.180**	**0.182**
	Mean - 1*SD	**0.000**	**0.059**	**0.036**	**0.024**	**0.021**	**0.014**	**0.017**
Steady-state	Mean + 1*SD	**0.000**	**0.125**	**0.255**	**0.295**	**0.309**	**0.318**	**0.321**
	Mean	**0.000**	**0.043**	**0.117**	**0.143**	**0.152**	**0.158**	**0.161**
	Mean - 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.001**
D: Allocation to T-bills								
Crash regime 1	Mean + 1*SD	**1.000**	**0.607**	**0.489**	**0.453**	**0.442**	**0.438**	**0.433**
	Mean	**0.996**	**0.349**	**0.275**	**0.250**	**0.240**	**0.238**	**0.233**
	Mean - 1*SD	**0.966**	**0.091**	**0.060**	**0.046**	**0.039**	**0.038**	**0.034**
Slow growth regime 2	Mean + 1*SD	0.083	**0.391**	**0.408**	**0.413**	**0.415**	**0.416**	**0.418**
	Mean	0.024	**0.202**	**0.217**	**0.221**	**0.223**	**0.224**	**0.225**
	Mean - 1*SD	0.000	**0.012**	**0.027**	**0.030**	**0.031**	**0.032**	**0.032**
Bull regime 3	Mean + 1*SD	0.000	**0.392**	**0.435**	**0.431**	**0.430**	**0.424**	**0.423**
	Mean	0.000	**0.225**	**0.249**	**0.240**	**0.237**	**0.229**	**0.229**
	Mean - 1*SD	0.000	**0.059**	**0.064**	**0.049**	**0.044**	**0.035**	**0.035**
Recovery regime 4	Mean + 1*SD	0.000	0.222	0.356	**0.385**	**0.396**	**0.401**	**0.402**
	Mean	0.000	0.090	0.178	**0.198**	**0.207**	**0.211**	**0.211**
	Mean - 1*SD	0.000	0.000	0.000	**0.012**	**0.019**	**0.022**	**0.019**
Steady-state	Mean + 1*SD	0.000	**0.347**	**0.410**	**0.418**	**0.421**	**0.420**	**0.419**
	Mean	0.000	**0.188**	**0.226**	**0.228**	**0.228**	**0.227**	**0.226**
	Mean - 1*SD	0.000	**0.030**	**0.043**	**0.038**	**0.036**	**0.033**	**0.033**
E: Overall Allocation to Stocks (Small and Large)								
Crash regime 1	Mean + 1*SD	**0.000**	0.478	0.701	**0.745**	**0.766**	**0.769**	**0.779**
	Mean	**0.000**	0.284	0.500	**0.545**	**0.565**	**0.569**	**0.576**
	Mean - 1*SD	**0.000**	0.091	0.299	**0.346**	**0.363**	**0.369**	**0.374**
Slow growth regime 2	Mean + 1*SD	**1.000**	**0.794**	**0.781**	**0.786**	**0.790**	**0.789**	**0.792**
	Mean	**0.893**	**0.590**	**0.586**	**0.591**	**0.593**	**0.592**	**0.593**
	Mean - 1*SD	**0.736**	**0.387**	**0.392**	**0.396**	**0.396**	**0.395**	**0.394**
Bull regime 3	Mean + 1*SD	**1.000**	**0.925**	**0.836**	**0.816**	**0.810**	**0.814**	**0.809**
	Mean	**1.000**	**0.760**	**0.651**	**0.625**	**0.616**	**0.617**	**0.611**
	Mean - 1*SD	**1.000**	**0.595**	**0.468**	**0.434**	**0.423**	**0.418**	**0.412**
Recovery regime 4	Mean + 1*SD	**1.000**	**0.872**	**0.808**	**0.802**	**0.799**	**0.804**	**0.805**
	Mean	**0.994**	**0.676**	**0.614**	**0.607**	**0.603**	**0.604**	**0.604**
	Mean - 1*SD	**0.962**	**0.481**	**0.421**	**0.411**	**0.407**	**0.404**	**0.403**
Steady-state	Mean + 1*SD	**1.000**	**0.926**	**0.839**	**0.817**	**0.809**	**0.808**	**0.807**
	Mean	**1.000**	**0.764**	**0.652**	**0.625**	**0.615**	**0.611**	**0.610**
	Mean - 1*SD	**1.000**	**0.602**	**0.466**	**0.433**	**0.420**	**0.414**	**0.411**

Table 6

Real-time Out-of-Sample Performance of Predictability Models

This table reports out-of-sample performance measures for portfolio choices under alternative return prediction models and for three investment horizons: 1, 12, and 120 months. $\mathbf{y}_t$ collects excess asset returns in the first n positions followed by *m* predictor variables. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The predictor is the dividend yield. For realized power utility (γ = 5), we report the negative of the calculated values. Investors aim at minimizing such values. In panels A and C, 'c.i.' stands for confidence interval. In panels B and D, positive differences reflect higher realized ex-post utilities for the MSIH(4,0) model. Panels A and B refer to the (pseudo) out-of-sample period, 1980:01-1999:12; panels C and D include the genuine out-of-sample period 2000:01-2003:12. In the table, MSIAH(*k*,*p*) denotes a *k*-state multivariate regime switching (MS), with shifts in intercepts (I), covariance matrices (H), and *p* autoregressive (A) lags.

	MSIAH(4,0)			**VAR(1)**			**MSIAH(4,1)**			**IID/Myopic**			**Min. Variance ptf.**			**Tangency ptf.**		
	T=1	**T=12**	**T=120**	**T=1**	**T=12**	**T=120**	**T=1**	**T=12**	**T=120**	**T=1**	**T=12**	**T=120**	**T=1**	**T=12**	**T=120**	**T=1**	**T=12**	**T=120**
	A – (Pseudo) Out-of-sample (1980:01 – 1999:12) realized power utility																	
Mean	0.248	**0.196**	**0.009**	**0.244**	0.198	0.021	0.247	0.209	0.034	0.246	0.212	0.028	0.246	0.207	0.012	0.245	0.197	0.011
St. deviation	0.048	0.091	0.004	0.032	0.083	0.015	0.038	0.082	0.028	0.026	0.087	0.011	0.024	0.074	0.005	0.047	0.106	0.006
5% c.i.-lower	0.243	0.168	0.007	0.240	0.174	0.017	0.243	0.185	0.017	0.243	0.183	0.022	0.243	0.180	0.010	0.239	0.161	0.008
5% c.i.-upper	0.255	0.225	0.011	0.248	0.223	0.025	0.252	0.233	0.051	0.249	0.241	0.034	0.249	0.231	0.015	0.251	0.231	0.014
10% c.i.-lower	0.243	0.173	0.007	0.241	0.178	0.018	0.243	0.189	0.019	0.243	0.187	0.023	0.244	0.184	0.010	0.240	0.166	0.008
10% c.i.-upper	0.253	0.220	0.011	0.248	0.218	0.025	0.251	0.230	0.049	0.249	0.236	0.033	0.249	0.227	0.015	0.250	0.225	0.014
	B – 100 × Differences in out-of-sample realized power utility vs. four-state regime switching model (1980:01 - 1999:12)																	
Mean	NA	NA	NA	-0.381	0.017	1.203	-0.115	1.270	2.331	-0.104	1. 460	1.764	-0.183	0.991	0.029	-0.288	0.058	0.036
St. deviation	NA	NA	NA	0.930	0.602	1.408	0.438	0.511	2.435	0.326	0.353	0.077	0.393	0.402	0.028	0.423	0.057	0.033
t-stat	NA	NA	NA	0.410	0.116	0.854	0.262	**2.489**	0.957	0.319	**4.115**	**2.306**	0.466	**2.479**	1.049	0.681	1.023	1.070
	C – Out-of-sample (2000:01 – 2003:12) realized power utility																	
Mean	**0.247**	0.420	NA	0.247	0.223	NA	0.247	**0.207**	NA	0.250	0.208	NA	0.249	0.241	NA	0.253	0.370	NA
St. deviation	0.046	0.236	NA	0.036	0.031	NA	0.039	0.053	NA	0.023	0.040	NA	0.018	0.034	NA	0.042	0.221	NA
5% c.i.-lower	0.243	0.377	NA	0.240	0.209	NA	0.243	0.197	NA	0.243	0.189	NA	0.243	0.217	NA	0.239	0.298	NA
5% c.i.-upper	0.255	0.616	NA	0.248	0.251	NA	0.252	0.220	NA	0.249	0.227	NA	0.249	0.256	NA	0.251	0.548	NA
10% c.i.-lower	0.243	0.400	NA	0.241	0.211	NA	0.243	0.199	NA	0.243	0.192	NA	0.244	0.219	NA	0.240	0.317	NA
10% c.i.-upper	0.253	0.602	NA	0.248	0.248	NA	0.251	0.218	NA	0.249	0.225	NA	0.249	0.252	NA	0.250	0.529	NA
	D – 100 × Differences in out-of-sample realized power utility vs. four-state regime switching model (2000:01 - 2003:12)																	
Mean	NA	NA	NA	0.000	-0.197	NA	0.001	-0.212	NA	0.003	-0.211	NA	0.001	-0.179	NA	0.006	-0.050	NA
St. deviation	NA	NA	NA	0.044	0.229	NA	0.038	0.234	NA	0.038	0.210	NA	0.048	0.243	NA	0.022	0.146	NA
t-stat	NA	NA	NA	0.002	-0.858	NA	0.010	-0.910	NA	0.083	-1.005	NA	0.025	-0.735	NA	0.253	-0.344	NA

Figure 1

Smoothed State Probabilities: Four-state model for Stock and Bond Returns

The graphs plot the smoothed probabilities of regimes 1-4 for the multivariate Markov Switching model comprising returns on large and small firms and 10-year bonds all in excess of the return on 30-day T-bills.

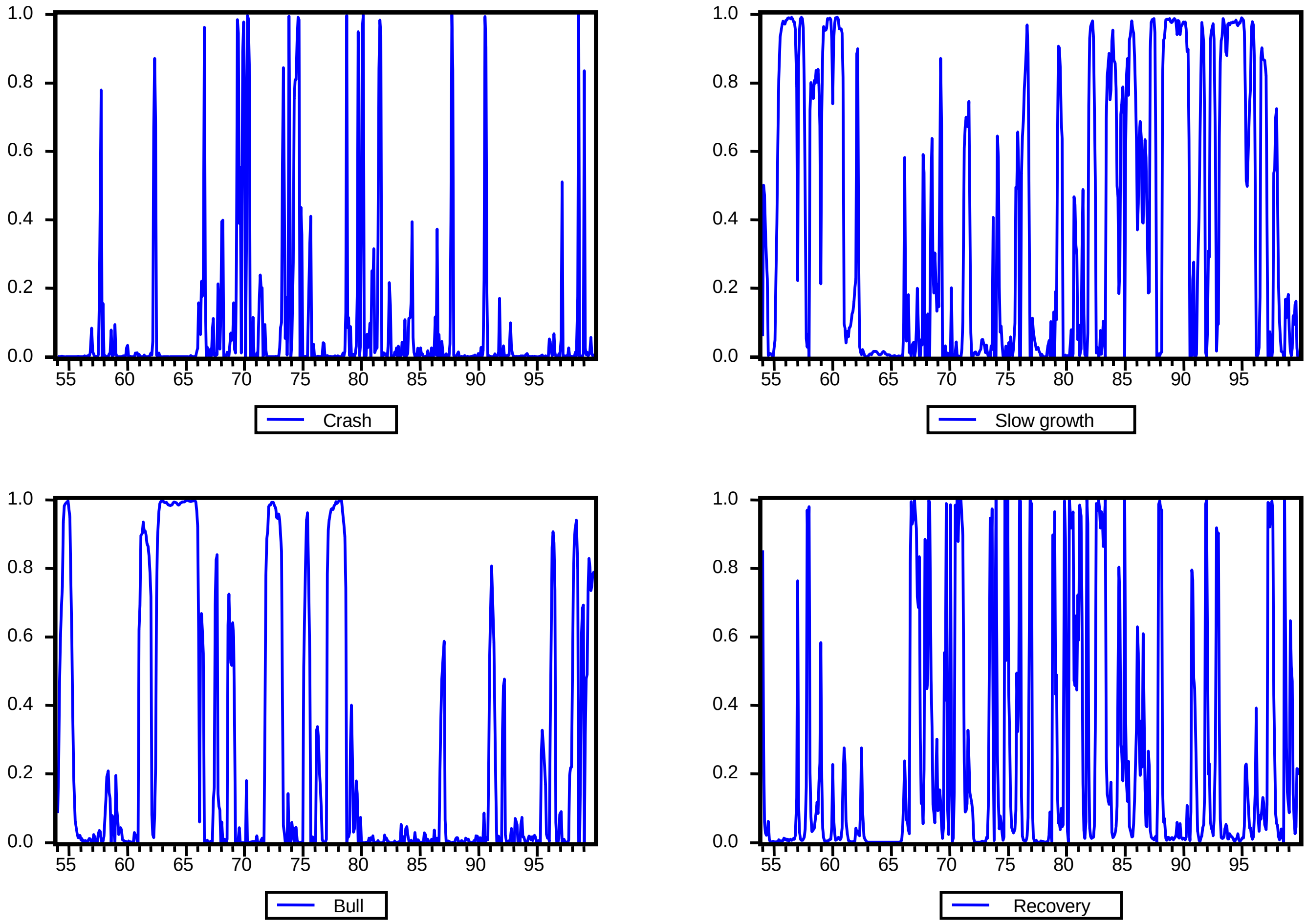

Figure 2

Optimal Portfolio Allocation for a Buy-and-Hold Investor

This graph plots the optimal allocation to stocks, bonds, and cash shown each quarter at various investment horizons. The plots assume the investor has power utility and coefficient of relative risk aversion γ = 5. Allocations are shown both for the four-state regime switching model and for a myopic investor who ignores the presence of regimes.

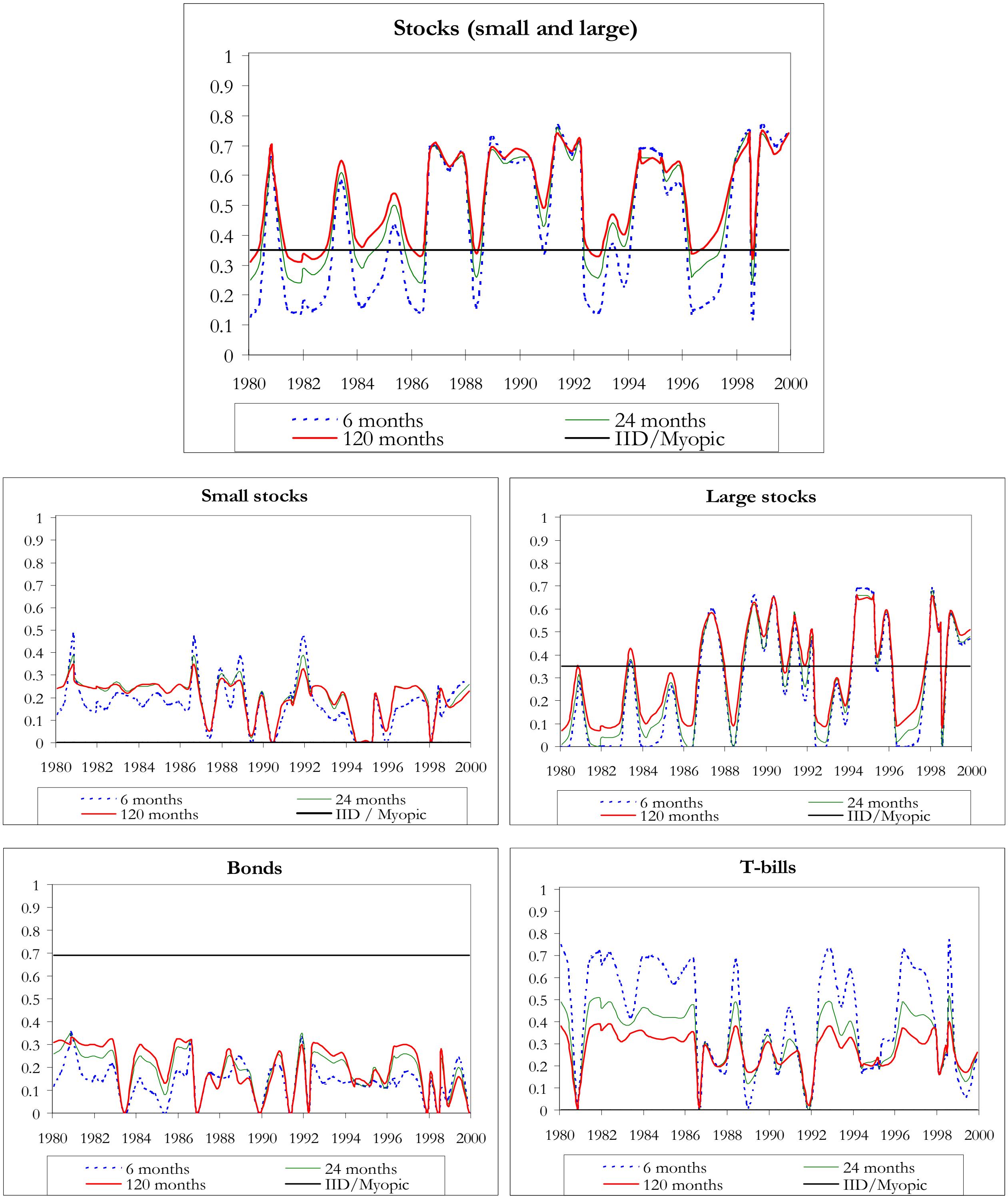

Figure 3

Optimal Portfolio Allocation as a Function of the Investment Horizon: Known Initial States

This graph varies the initial state probabilities perceived by the investor and traces out the resulting asset allocation. The graphs show the optimal allocation to four asset classes — small and large stocks, long-term bonds, and T-bills — as a function of the investment horizon for a buy-and-hold investor with constant relative risk aversion $\gamma = 5$.

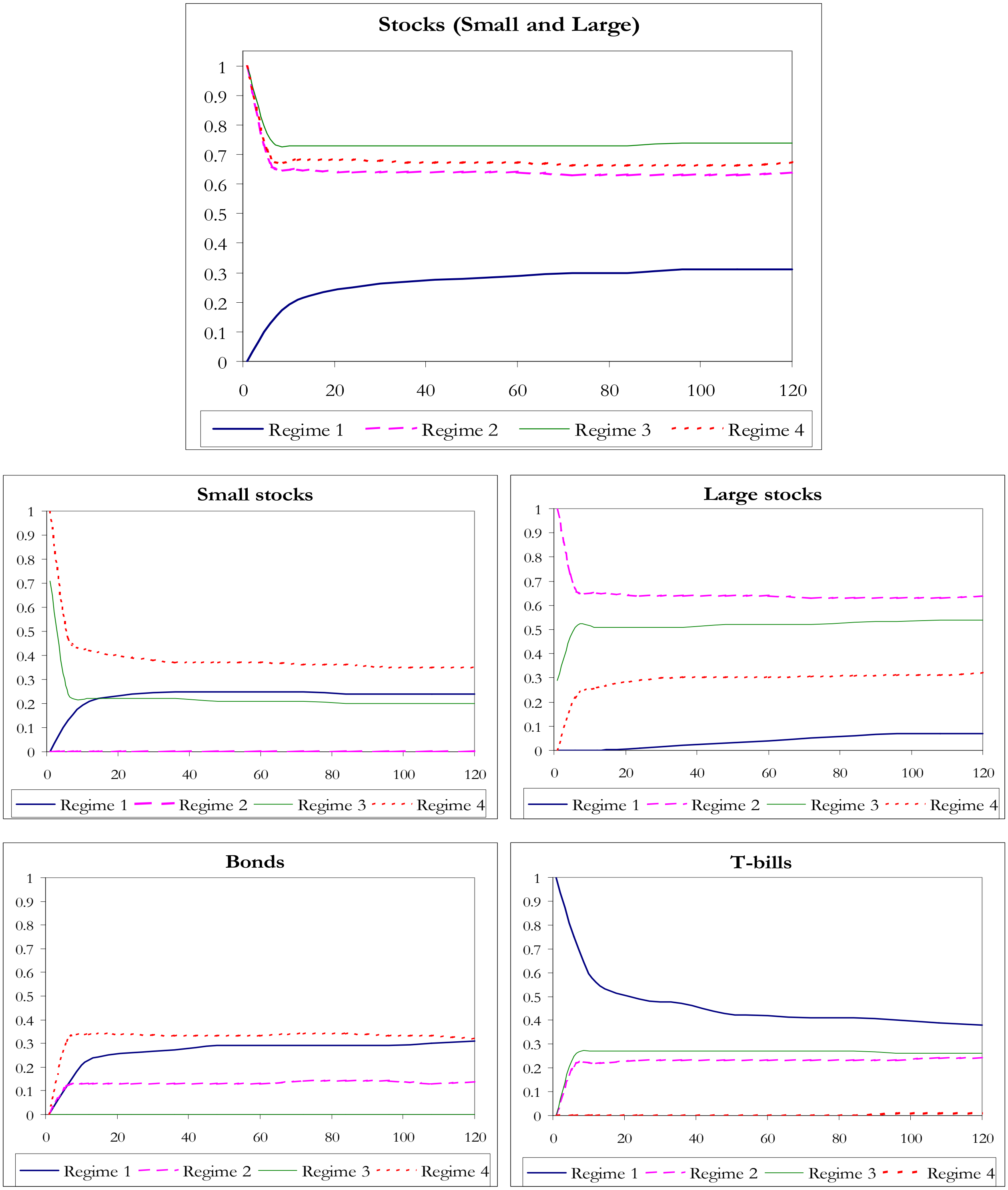

Figure 4

Effect of Uncertain States on Asset Allocation

This Figure considers the case with uncertainty about the current regime. The graphs show the optimal allocation to four asset classes — small and large caps, long-term bonds, and 1-month T-bills — as a function of the investment horizon for a buy-and-hold investor with constant relative risk aversion $\gamma = 5$.

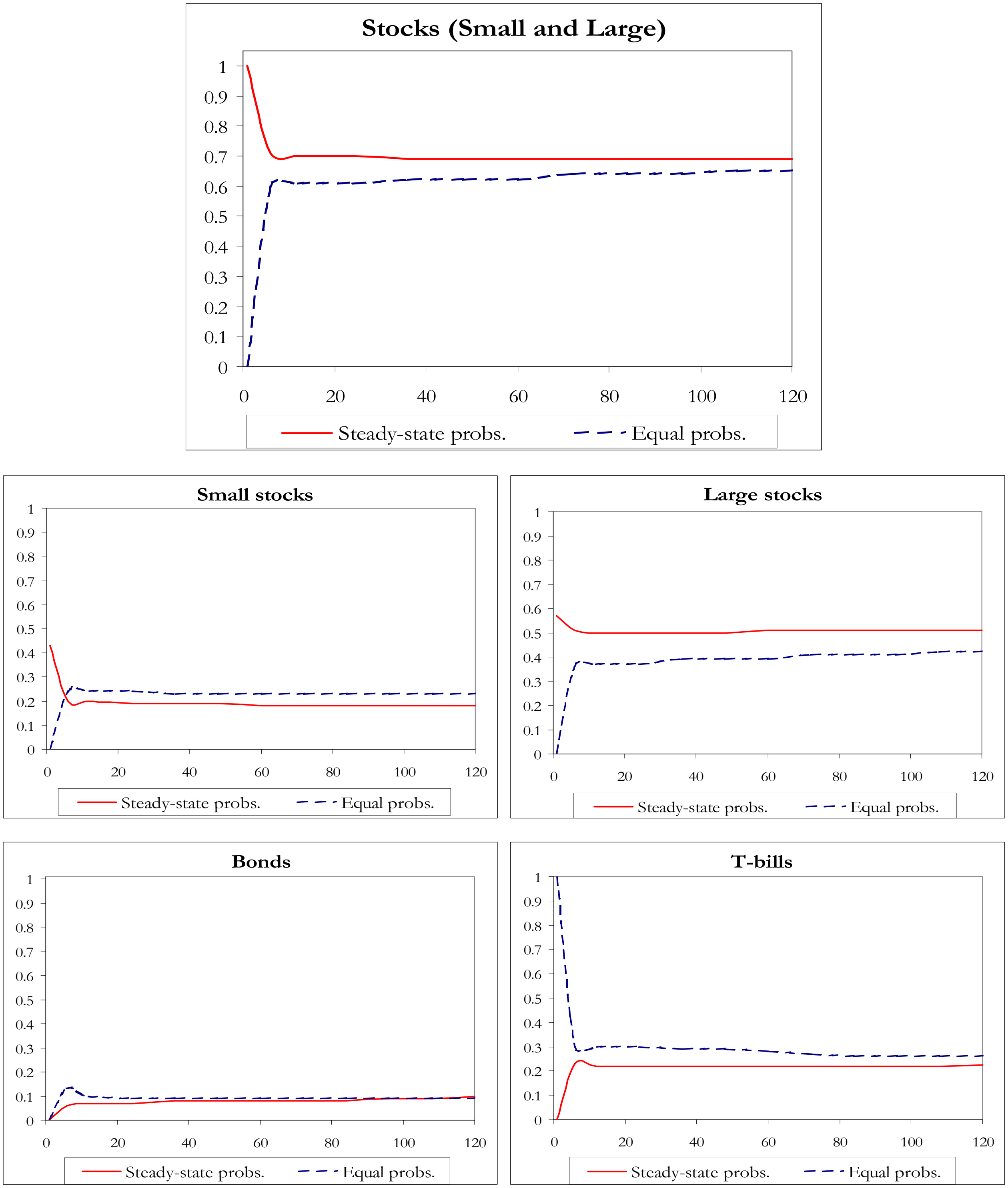

Figure 5

Effect of Risk Aversion on Asset Allocation

Plots of the optimal allocation to stocks and long-term bonds at three investment horizons. The plots assume the buy-and-hold investor has power utility and coefficient of relative risk aversion γ in the interval [1, 50]. Allocations are shown both for the four-state regime switching model and for a myopic investor who ignores the presence of regimes.

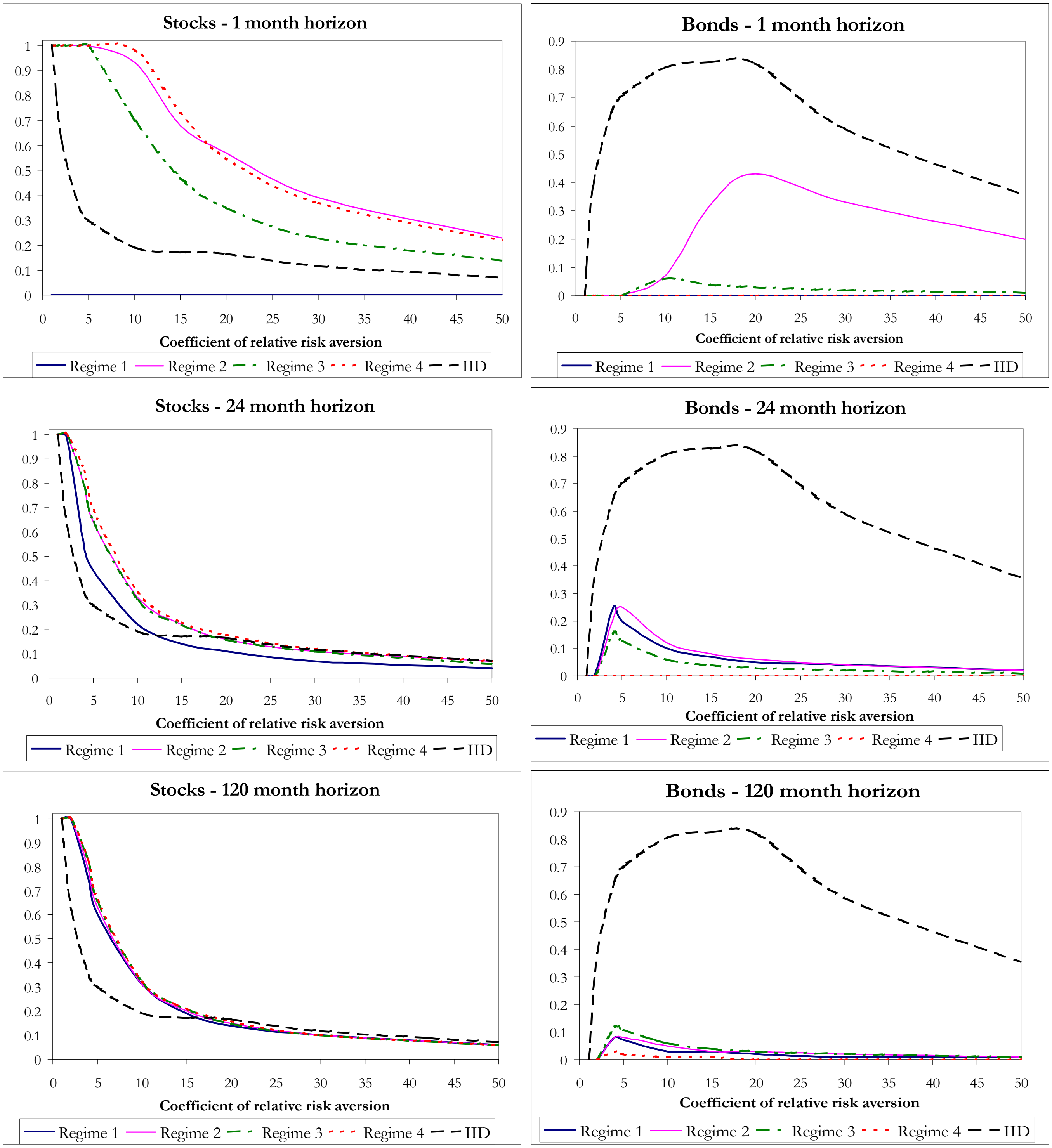

Figure 6

Optimal Asset Allocation at Different Values of the Dividend Yield – Linear Model

For each asset class, the graphs plot the optimal allocation as a function of the investment horizon and the dividend yield level for an investor with constant relative risk aversion γ = 5. Asset returns and the dividend yield follow a single-state VAR(1) model.

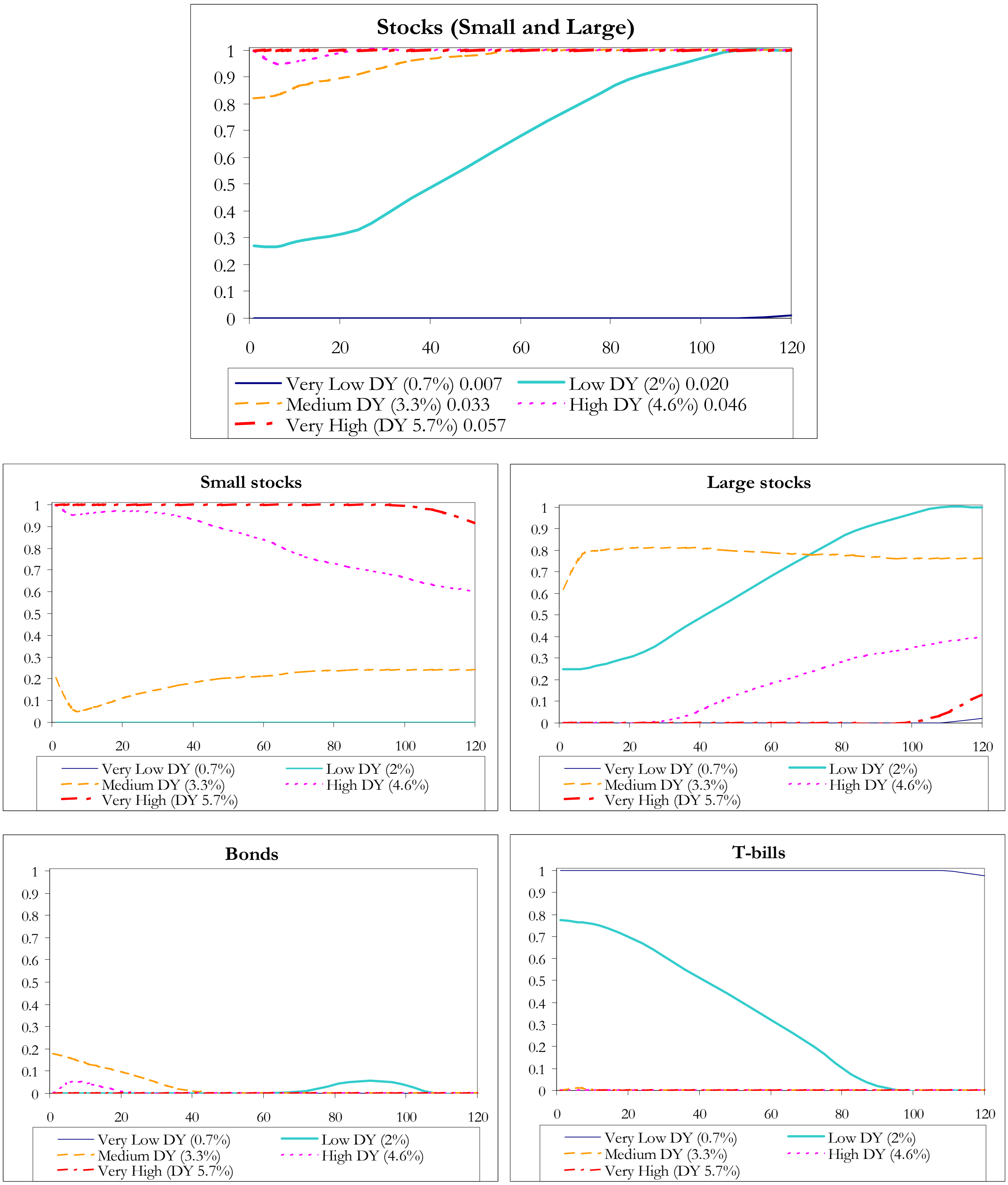

Figure 7

Predictability from the Dividend Yield

The graphs plot the optimal allocation as a function of the investment horizon for an investor with constant relative risk aversion $\gamma = 5$ for six configurations of initial state probabilities: certainty of being in regimes 1-4, equal state probabilities, and ergodic state probabilities. In each graph, the dividend yield is set at its unconditional sample mean.

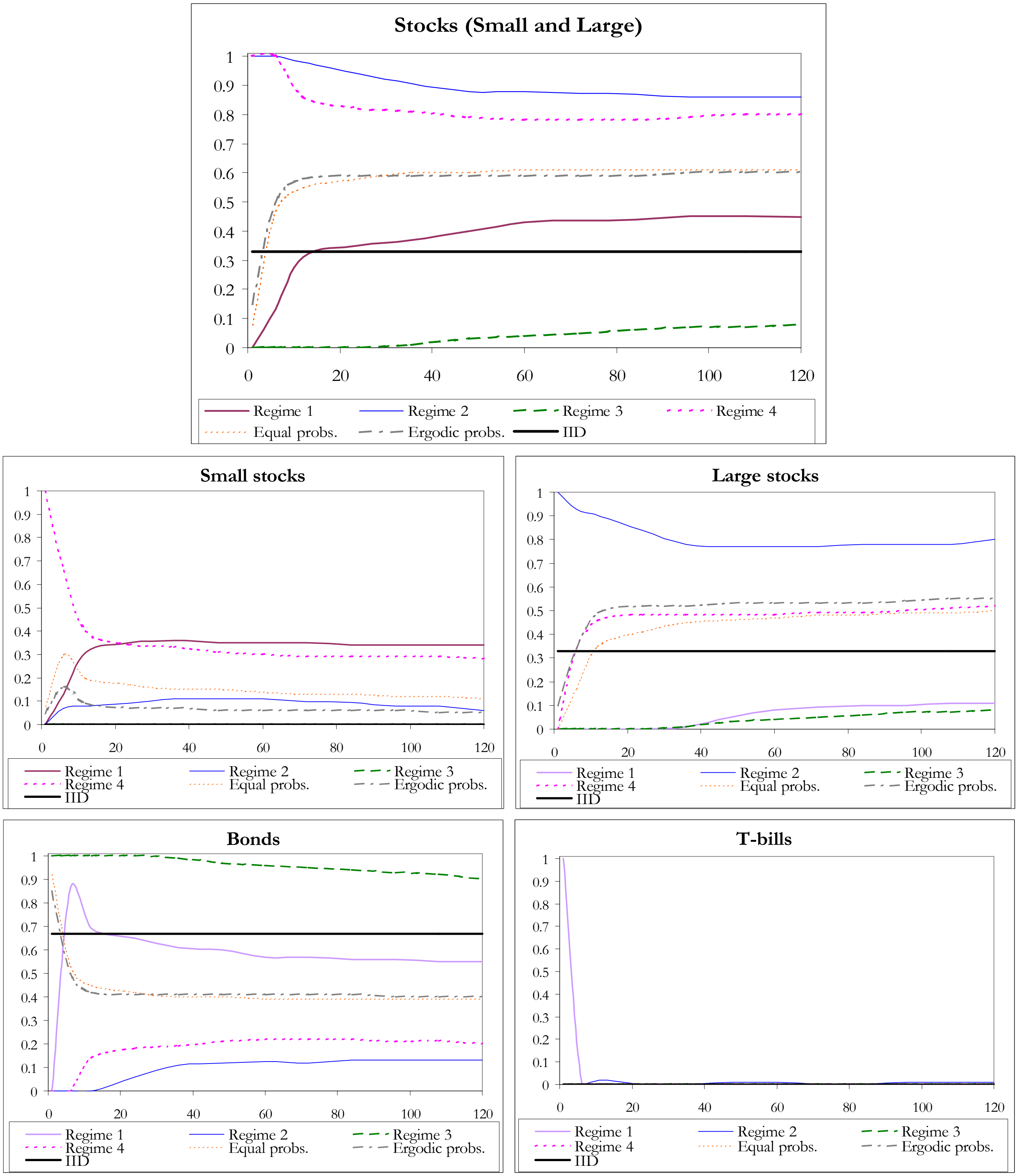

Figure 8

Optimal Asset Allocation at Different Values of the Dividend Yield under Regime Switching

These graphs plot the optimal asset allocation as a function of the investment horizon and the value of the dividend yield for an investor with constant relative risk aversion $\gamma = 5$. The perceived state probabilities are fixed at their ergodic, full-sample values.

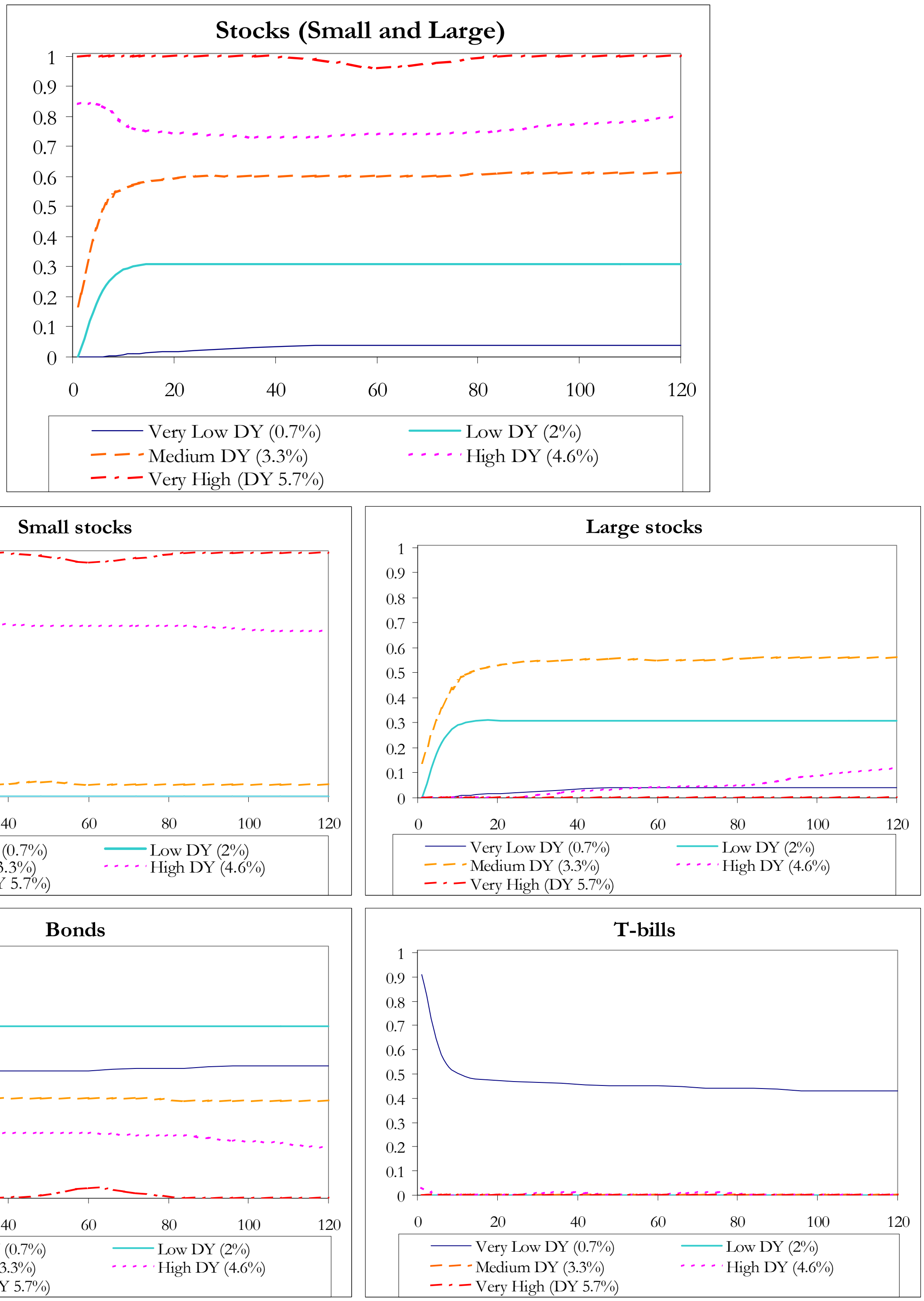

Figure 9

Volatility and Sharpe Ratios as a Function of the Investment Horizon

These graphs plot monthly volatility and Sharpe ratios of returns on each asset class under three alternative models (four-state, MSIH(4,0), four-state VAR(1) model with predictability from the dividend yield, MSIAH(4,1), single-state model with predictability from the dividend yield, VAR(1)). State probabilities and the dividend yield are set at their steady-state values. Values are normalized by the square root of the horizon so the IID model corresponds to horizontal lines.

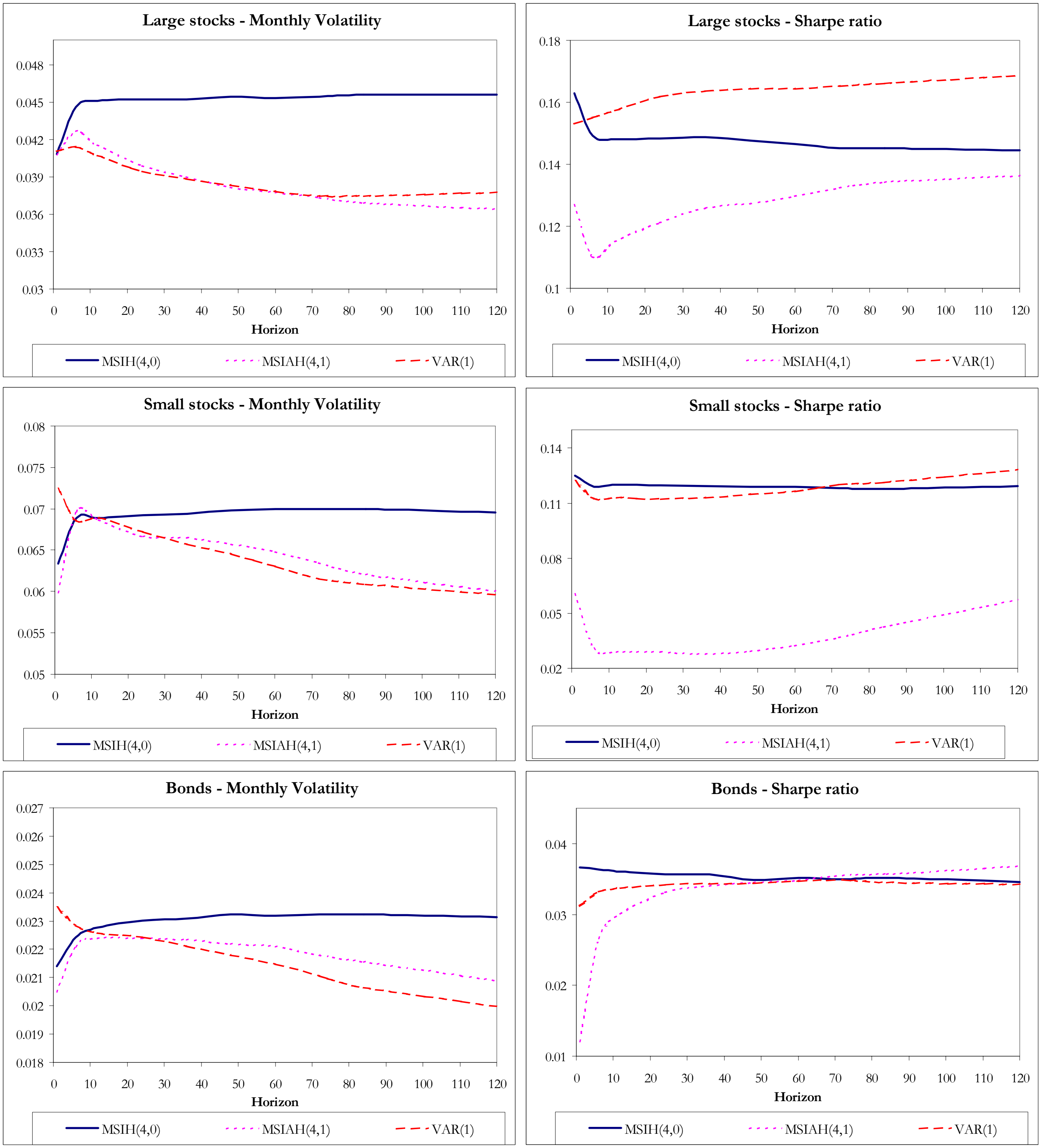

Figure 10

Optimal Consumption-Wealth Ratio

The graph plots the optimal consumption-wealth ratio as a function of the investment horizon for the four-state regime switching model assuming power utility and relative risk aversion coefficient $\gamma = 5$ and (annualized) rate of subjective time preference of 5%. The investor consumes only at time t and at time t + T. As a benchmark, the solid line reports the consumption-wealth ratio when the regime probabilities are set at their steady-state values.

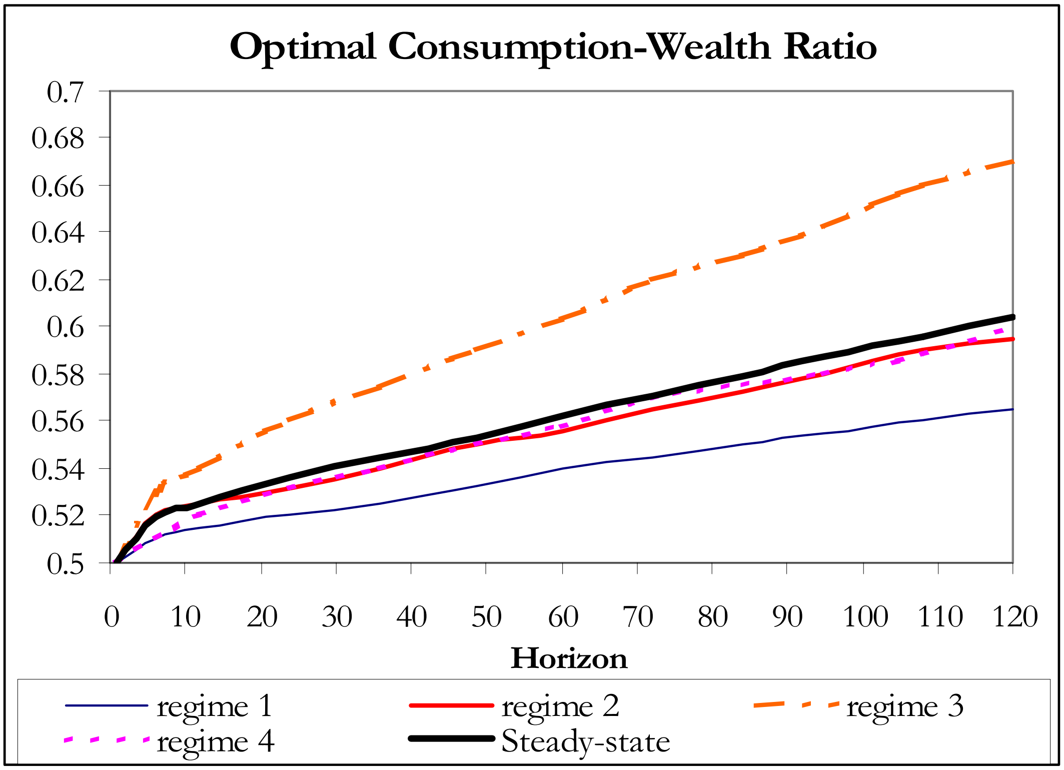

Figure 11

Utility Costs from Ignoring Regimes

The graph plots the compensation (as an annualized percentage) required to persuade a buy and hold investor with power utility (and $\gamma = 5$) to be willing to ignore regimes in asset returns, starting from steady-state probabilities.

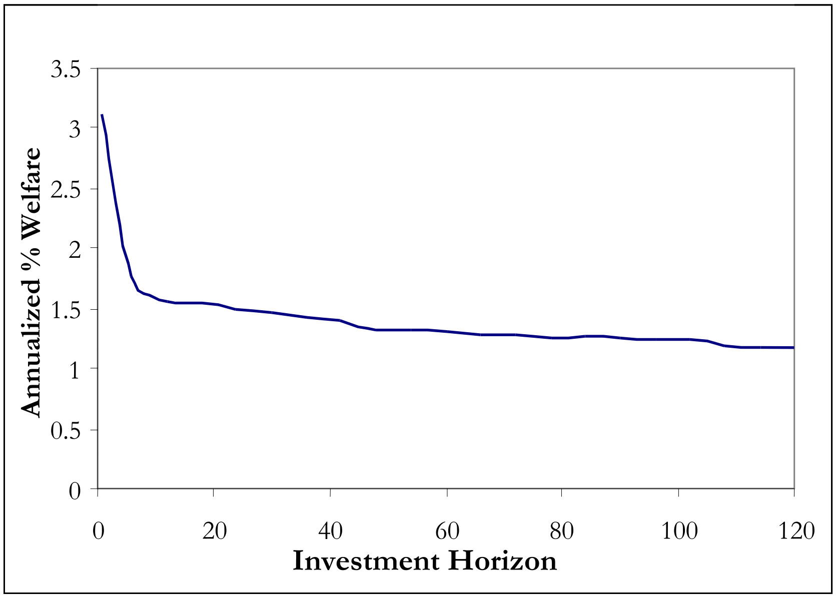

Figure 12

90% Bootstrapped Confidence Bands for Utility Costs from Ignoring Regimes

The graphs plot means and bootstrap confidence intervals for the compensation (as a fraction of initial wealth) required to persuade a buy-and-hold investor with power utility (and $\gamma = 5$) to be willing to ignore regimes in asset returns. State probabilities are set at their steady-state values.

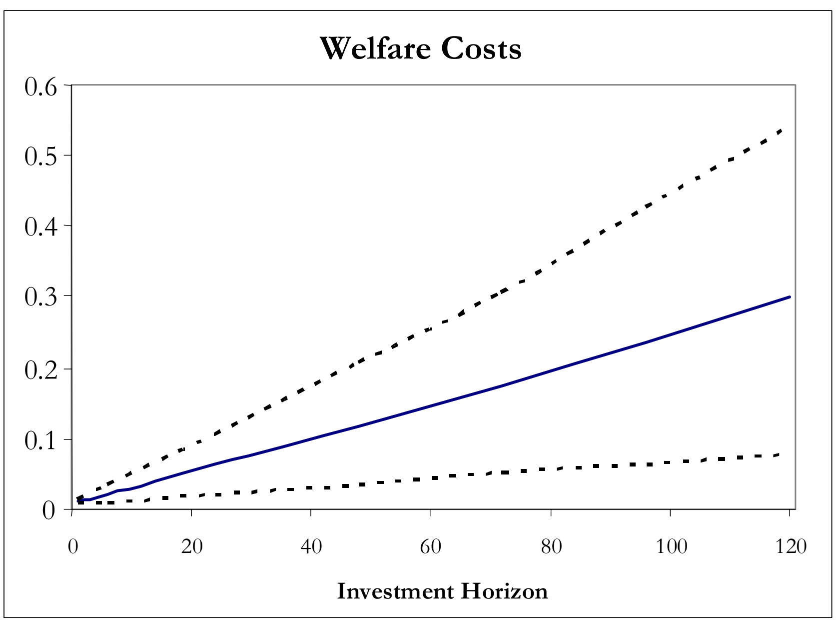

Figure B1

Sampling Error in Monte Carlo Approximation of Expected Utility

These graphs plot the optimal portfolio weights for a buy and hold investor with power utility (and $\gamma = 5$) when the number of Monte Carlo simulations used to approximate expected utility is increased from 2,000 to 50,000, in steps of 2,000.