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A General Affine Earnings Valuation Model*

ANDREW ANG aa610@columbia.edu
Columbia University and NBER, 3022 Broadway, 805 Uris, New York, NY 10027

JUN LIU jliu@anderson.ucla.edu
UCLA, Anderson School C509, UCLA, CA 90095

Abstract. We introduce a methodology, with two applications, that incorporates stochastic interest rates, heteroskedasticity and risk aversion into the residual income model. In the first application, goodwill is an affine (constant plus linear term) function where the constant and linear coefficients are time-varying. Homoskedastic risk gives rise to a constant risk premium, while heteroskedastic risk gives rise to linear state-dependent risk premiums. In the second application, we present a class of models where a non-linear function for the price-to-book ratio can be derived. We show how interest rates, risk, profitability and growth affect the price-to-book ratio.

Keywords: stock valuation, earnings, residual income model, asset-pricing, affine model, linear information dynamics

This paper provides a parametric class of models that shows how a firm's market value relates to accounting data under stochastic interest rates, heteroskedasticity and adjustments for risk aversion. We use the framework of the Residual Income Model (RIM), which expresses the value of a stock as the firm's book value plus the expected future discounted value of the firm's abnormal (or residual) earnings. Our methodology builds on the framework of Feltham and Ohlson (1999), who extend the RIM to a no-arbitrage setting to accommodate time-varying interest rates and risk aversion. Feltham and Ohlson give a partial parametric model of stock valuation using accounting information in this setting. In a heteroskedastic environment with stochastic interest rates and risk aversion, we extend this analysis in several ways. First, we apply this methodology to the case where the dynamics of accounting variables are expressed in dollar amounts as in Feltham and Ohlson (1995). Second, we derive a solution for the price-to-book ratio of a firm as a function of stochastic interest rates, accounting rates of return and growth in book.

Our first result extends the Linear Information Model (LIM) developed in Ohlson (1995) and Feltham and Ohlson (1995). The LIM presents firm value as a linear function of current observable accounting information and is derived under constant discount rates. This assumption leads to a standard simplification where a single discount factor can be applied to all future periods. The discount factor can incorporate an *ad hoc* adjustment for risk. There are certain questions, however, which cannot be addressed under this assumption. For instance, is it always possible to incorporate risk aversion as a spread in a constant

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discount factor? Can a linear solution be found under the addition of time-varying interest rates, heteroskedasticity and risk aversion? Feltham and Ohlson (1999) show how to adjust the RIM for risk but do not provide a complete parametric model to answer these questions directly. They hint, however, that a model as tractable as the LIM might exist under more general conditions. We propose a class of models where an extended Feltham-Ohlson linear form can be preserved under stochastic interest rates and time-varying risk premiums. Under risk neutrality and constant interest rates, our model reduces to the Feltham-Ohlson LIM.

Our extension to the LIM expresses firm value as an affine combination (constant plus linear form) of abnormal earnings and book value. We show that it is possible to choose a parameterization so that an affine form for goodwill (the difference between price and book value) exists under stochastic interest rates and risk aversion, provided that the interest rate process is uncorrelated with accounting variables. The coefficients in the affine form are time-varying, and reflect the dependence on current zero coupon bond prices. Risk aversion potentially affects both the constant and the linear terms. With risk aversion, homoskedasticity is captured conveniently only in the constant term, while risk aversion combined with heteroskedastic risk leads to state-dependent risk premiums which appear in the linear terms.

Under the LIM, dollar amounts of residual earnings are assumed to follow a stationary process and firm value is a linear function of contemporaneous earnings information. Questions about the rate of earnings growth rather than the dollar amount of earnings, or firm growth, cannot easily be addressed in this setting. Our second application focuses on the price-to-book ratio, rather on than the dollar difference in the price and book, as in the LIM. We apply our pricing methodology to the RIM framework of Feltham and Ohlson (1999), with a normalization by book-value.[1] This framework models the price-to-book as a function of stochastic interest rates, a rate of return measure based on profitability (accounting returns of earnings in excess of the risk-free rate), and firm growth. Risk aversion and heteroskedasticity of all three variables are explicitly modeled.

Ratio analysis highlights the effect of the rate of profitability and growth on valuation. We might anticipate that higher profitability of a firm would lead to higher price-to-book ratios, as higher profitability increases firm value relative to current book value. However, the effect of growth in book on the price-to-book is not so clear.[2] By directly parameterizing growth in book we can determine how the price-to-book ratio behaves when parameters underlying the process for growth in book are changed. Another comparative static of interest is the mean-reversion of profitability and growth. Standard economic arguments in a competitive environment argue that these variables are mean-reverting.[3] Does higher mean-reversion in profitability or growth lead to higher or lower price-to-book ratios? Finally, risk-averse agents are affected by the volatility of interest rates, profitability and growth. Hence, under risk aversion, volatility of these variables affects the price-to-book.

We present a closed-form non-linear solution of the price-to-book ratio. The solution formula allows characteristics of the behavior of the price-to-book to be examined by comparative statics. As expected, increasing profitability increases the price-to-book ratio. Increasing the conditional mean of book growth unambiguously increases the price-to-book ratio. We find that increasing the autocorrelation of profitability or growth in book increases the price-to-book. That is, *ceteris paribus,* firms with highly mean-reverting profitability or growth have lower price-to-book ratios.

Conducting comparative statics with respect to the dynamics of the interest rate yields one surprising behavior of the price-to-book which seems counter-intuitive. We may expect that increasing the volatility of interest rates would decrease the price-to-book, since nominal interest rates are positive (which is the case under a Cox-Ingersoll-Ross (1985) term structure model) and increasing the volatility of interest rates implies higher discount factors in future periods. However, under conservative accounting and risk neutrality, the price-to-book is an increasing function of the volatility of all state variables including, surprisingly, the volatility of the interest rate. This counter-intuitive behavior is due to a Jensen's inequality effect which dominates under risk neutrality. When risk aversion is introduced, the price-to-book ratio may fall as the volatility of the interest rate increases.

Our methodology incorporates several stylized facts of interest rates and risk aversion which bear on accounting valuation. First, interest rates are time-varying, which affects the discount factors used in future periods. This is a "denominator" effect but interest rates also predict future abnormal earnings, which is a "numerator" effect.[4] Our formal methodology simultaneously handles both the denominator and numerator effect of time-varying interest rates. Moreover, predictability of accounting information by any variable, not just interest rates, can be accommodated. Second, risk-averse agents in the economy need to be compensated for the uncertainty in the evolution of financial statement information because accounting information is a driving factor of prices. Viewed another way, the risk premiums associated with the uncertainties of accounting information are reflected in discount factors, as discussed by Feltham and Ohlson (1999). Our methodology tractably incorporates risk aversion and shows how it affects the LIM and book-to-market ratio dynamics.

Our methodology tractably and parsimonously incorporates rich dynamics of interest rates and accounting variables by using "affine" processes (Duffie and Kan, 1996), where both the conditional mean and conditional volatility take on affine forms (constant plus linear terms). Ohlson (1995) and Feltham and Ohlson (1995) rely on simple AR(1) or modified AR(1) processes to derive the LIM. These are special cases of the affine set-up. The affine processes also formally encompass Feltham and Ohlson (1999)'s partial model, since they can incorporate both heteroskedasticity of the driving variables and risk adjustments.

The rest of the paper is organized as follows. In Section 1 we describe the role of a "pricing kernel" in no-arbitrage valuation. Section 2 presents the affine extension of the LIM and shows that linearity can survive the introduction of stochastic interest rates and risk aversion. Section 3 applies the methodology to the case of ratio dynamics, and presents a closed-form model of the price-to-book ratio. Comparative static exercises show how changes in the interest rate process, profitability, growth and risk aversion affect the price-to-book of the firm. Section 4 concludes.

1. A Parameterization of the No-Arbitrage RIM

This section introduces notation to interpret no-arbitrage valuation. This is accomplished by specifying a tractable "pricing kernel," which we parameterize as a log-affine form in Section 1.1. Similar to Feltham and Ohlson (1999)'s analysis, we bring the RIM into this framework in Section 1.2.

1.1. A Log-Affine Pricing Kernel

The assumption of no-arbitrage, together with some technical conditions (see Harrison and Kreps, 1979), guarantees the existence of a random process which prices all assets in the economy. This random process is called a "pricing kernel," which we denote by π_{t+1}. The pricing kernel is unique if markets are complete, meaning that the number of securities is sufficient to insure against all possible sources of risk in the economy. While the assumption of no-arbitrage guarantees the existence of π_{t+1}, no-arbitrage gives no information, however, about the true functional form of π_{t+1}, only that some π_{t+1} exists.[5]

The pricing kernel π_{t+1} relates the price of a security today with its payoffs in the next period. For any asset:[6]

$$P_t = \mathrm{E}_t[\pi_{t+1} Z_{t+1}], \tag{1}$$

where P_t is the price of an asset, and Z_{t+1} are its payoffs at time $t+1$. Note that if $Z_{t+1} = 1$, a unit payoff, then P_t is the price of a one period bond. In the case of a stock S_t, the price of the stock is related to its dividends δ_t by:

$$S_t = \mathrm{E}_t[\pi_{t+1}(S_{t+1} + \delta_{t+1})]. \tag{2}$$

Time-varying interest rates and risk aversion are captured by the pricing kernel π_{t+1}. To separate out the role of the short rate r_t and risk in π_{t+1}, we introduce another random variable ξ_{t+1} which we define as:

$$\xi_{t+1} = \frac{\pi_{t+1}}{\mathrm{E}_t(\pi_{t+1})}. \tag{3}$$

Recall that $\mathrm{E}_t(\pi_{t+1}) = \exp(-r_t)$ is the price of the one-period risk-free bond at time t. This enables us to rewrite equation (2) as:

$$S_t = \mathrm{E}_t[\exp(-r_t)\xi_{t+1}(S_{t+1} + \delta_{t+1})], \tag{4}$$

where $\pi_{t+1} = \exp(-r_t)\xi_{t+1}$. Equation (4) separately decomposes the role of the pricing kernel into the short rate process r_t, and ξ_{t+1}, which allows for explicit adjustments for risk aversion in the no-arbitrage environment.

The most general parameterization of no-arbitrage is determined by (i) $\xi_{t+1} > 0$ and (ii) $\mathrm{E}_t(\xi_{t+1}) = 1$. We now assume a parameterization for ξ_{t+1}. Specifically, we assume that ξ_{t+1} is log-normally distributed:

$$\xi_{t+1} = \exp\left(-\tfrac{1}{2}\gamma'\sigma_t\sigma_t'\gamma + \gamma'\sigma_t\epsilon_{t+1}\right). \tag{5}$$

where ϵ_{t+1} is a $K \times 1$ vector IID N(0,I), γ is a $K \times 1$ vector and σ_t is a $K \times K$ matrix. The subscript t on σ_t indicates that σ_t may be a function of time t information, and hence may vary through time. The errors ϵ_{t+1} represent all shocks to K driving variables in the economy. For now, these driving variables remain unspecified, but they can be any variable which affects prices in the economy. In Section 2 we specify the driving variables to be accounting variables in levels and in Section 3 we specify the driving variables to be ratios.

The log-normal pricing kernel in equation (5) is a valid pricing kernel. First, it satisfies strict positivity as $\exp(\cdot) > 0$. Second, by property of the log-normal distribution (see Appendix), $\mathrm{E}_t[\xi_{t+1}] = 1$.[7]

We refer to γ as the price of risk which captures the risk aversion of agents. The following example illustrates the role of γ. Let time $t = 0$, and suppose, without loss of generality, the prevailing short rate $r_0 = 0$. There is only one source of risk in the economy, say from earnings, so $K = 1$, and $\epsilon_1 \sim N(0, 1)$. We would like to price a claim which has a payoff $\sigma_0\epsilon_1$, that is the security's payoff is the unanticipated factor shock. The price of this security P_0 is given by:

$$\begin{aligned} P_0 &= \mathrm{E}_0[\xi_1\sigma_0\epsilon_1] \\ &= \mathrm{E}_0\left[\exp\left(-\tfrac{1}{2}\gamma^2\sigma_0^2 + \gamma\sigma_0\epsilon_1\right)\sigma_0\epsilon_1\right] \\ &= \gamma\sigma_0^2. \end{aligned} \tag{6}$$

The last equality can be derived using a lemma in the Appendix. Under the case of risk neutrality, $\gamma = 0$ and the price of the security is zero. This is expected, because under risk neutrality the price of the security is just the expected value of the security's payoffs, which is zero. Under risk aversion $\gamma \neq 0$, and risk-averse agents must be compensated to take on a risk with a zero expected value payoff. If $\gamma < 0$, the price of the security is negative, which is less than the risk-neutral price of zero. That is, risk-averse agents must be paid to bear this risk. The greater the degree of risk aversion, the more negative the value of γ, and the more risk averse agents must be compensated for bearing risk.

Another interpretation of the role of γ is to look at the role risk plays in factor pricing. Risk is related to the degree of correlation between shocks in the driving variables (factors) and the negative of the pricing kernel. To make this concrete, again suppose that there is only one driving variable in the economy, $K = 1$, time $t = 0$, and $\epsilon_1 \sim N(0, 1)$. Using the lemma stated in the Appendix, the conditional covariance of ϵ_1 with $-\xi_1$ at time $t = 0$ is given by:

$$\begin{aligned} \mathrm{cov}_0(\epsilon_1, -\xi_1) &= \mathrm{cov}_0\left(\epsilon_1, -\exp\left(-\tfrac{1}{2}\gamma^2\sigma_0^2 + \gamma\sigma_0\epsilon_1\right)\right) \\ &= -\gamma\sigma_0^2. \end{aligned} \tag{7}$$

If factor shocks have a correlation of zero, then their prices of risk are zero. If factor shocks have non-zero correlation with the pricing kernel, then the degree of correlation is a measure of the degree of risk associated with that variable. Only if γ is non-zero will the covariance of the factor shock with the signed normalized pricing kernel ξ_1 be non-zero. In particular, if γ is negative, the covariance with $-\xi_1$ is positive, which translates to non-zero risk aversion. As the degree of risk aversion becomes greater, γ becomes more negative, and the correlation between the driving factor and $-\xi_1$ becomes more positive.

1.2. The RIM and No-Arbitrage Valuation

We now introduce notation to be used for applying the log-normal parameterization of the pricing kernel to level and ratio analysis. This notation enables the role of ξ_{t+1} to be used in evaluating the expectations of future periods. We start by working with pricing kernel π_{t+1} notation, and then we define notation so that expectations of future periods can be taken with respect to ξ_{t+1}. These transformed expectations are equivalent to the pricing kernel expectations.

The pricing kernel π_{t+1} relates the price of a stock to its future payoffs. Repeating equation (2) we have:

$$S_t = \mathrm{E}_t[\pi_{t+1}(S_{t+1} + \delta_{t+1})].$$

Iterating this forward and assuming transversality $\mathrm{E}_t(\pi_\tau S_\tau) \to 0$ as $\tau \to \infty$, we get the Dividend Discount Model (DDM):

$$S_t = \mathrm{E}_t\left[\sum_{i=1}^{\infty} \pi_{t+i}\delta_{t+i}\right]. \tag{8}$$

To relate the dividend process back to accounting variables, residual accounting makes the clean surplus accounting assumption:

$$y_t = y_{t-1} + x_t - \delta_t \tag{9}$$

where y_t is the book value of equity and x_t represents net earnings at time t. This says that the increase in the book value of equity comes from earnings less dividends paid.

Feltham and Ohlson (1999) develop a generalized RIM under time-varying interest rates and risk aversion. They show that by using clean surplus accounting and the DDM in equation (8), the equity value of a firm can be written as:[8]

$$S_t = y_t + \mathrm{E}_t\left[\sum_{i=1}^{\infty} \pi_{t+i} x^a_{t+i}\right] \tag{10}$$

where x^a_t are abnormal earnings:

$$x^a_t = x_t - (\exp(r_{t-1}) - 1)y_{t-1} \tag{11}$$

where r_{t-1} is the risk-free short rate from time $t-1$ to time t, which can be stochastic. In the basic RIM with constant short rates, the appropriate capital charge is the (constant) risk-free rate. In a setting with stochastic short rates, the relevant capital charge component of abnormal earnings is the riskless one-period interest rate applied to the book value at the start of the period. Note that risk is embedded in the Feltham and Ohlson (1999) framework by using the pricing kernel to take the future expectations of equation (10).

We now re-write equation (10) so that the expectation is taken with respect to ξ_{t+1}, called the "risk-neutral measure." Part of the role of ξ_{t+1} in the pricing kernel is to separate the effects of the risk-free rate and risk aversion in π_{t+1}. Repeating equation (4) we have:

$$S_t = \mathrm{E}_t[\exp(-r_t)\xi_{t+1}(S_{t+1} + \delta_{t+1})].$$

This expectation is taken under the "real measure" (the probability density function existing in the real world). It can be rewritten as:

$$S_t = \mathrm{E}_t^Q[\exp(-r_t)(S_{t+1} + \delta_{t+1})]. \tag{12}$$

This expectation is taken under the measure, or probability density function, Q. The measure Q is called the risk-neutral measure because under Q the price of equity is just the expected discounted payoffs of the security. This is also called the equivalent martingale measure.[9]

The pricing kernel π_{t+1} and ξ_{t+1} are related recursively by:

$$\frac{\pi_t}{\pi_0} = \prod_{j=0}^{t} \xi_j \exp(-r_{j-1}). \tag{13}$$

Using the risk-neutral measure we can equivalently rewrite the value of equity using the RIM in equation (10), by substituting for the pricing kernel in the infinite sum, as:

$$S_t = y_t + \sum_{i=1}^{\infty} \mathrm{E}_t^Q \left[\left(\prod_{j=0}^{i-1} \exp(-r_{t+j}) \right) x_{t+i}^a \right] \tag{14}$$

This is an equivalent representation of the RIM valuation equation.

The advantage of the parameterization of equation (14) is that it explicitly shows the interaction of stochastic short rates (through the stochastic discount factor $\prod \exp(-r_{t+j})$) and risk aversion (through the density ξ_t used to evaluate the expectation). If there is no risk, then $\gamma = 0$ so $\xi_t = 1$ and the real and risk-neutral measure coincide.

The special case of the RIM presented in Ohlson (1995) shows the simplifications which arise using constant discount rates in the valuation problem. The following claim shows how the RIM with constant interest rates and a constant *ad hoc* risk adjustment is a special case of our general valuation methodology.

Claim 1.1 *Assume that*

1. *short rates are constant, so* $\exp(r_t) = R_f$
2. *the risk premium is constant,* $cov_t\,(r_{t+1}^s, -\xi_t) = \bar{\sigma}$, *where* r_s *is the return on the stock* $r_{t+1}^s \equiv (S_{t+1} + \delta_{t+1})/S_t$

Then the traditional RIM of stock valuation holds:

$$S_t = y_t + \sum_{i=1}^{\infty} R^{-i} \mathrm{E}_t \left[x_{t+i}^a \right] \tag{15}$$

where S_t *is the value of firm equity,* $R = R_f + \bar{\sigma}$, y_t *is book value, and* x_t^a *are abnormal earnings:*

$$x_t^a = x_t - (R - 1) y_{t-1}$$

where x_t *represents firm earnings.*

Compared to the traditional RIM, a pricing kernel methodology must be employed once interest rates are stochastic and risk premiums are time-varying. Under this generalized setting the valuation problem becomes much more complex. Since the discount factor R is constant in Claim 1.1, it passes through the expectation operator in equation (14). When short rates are stochastic, the discount factor $\prod \exp(-r_{t+j})$ is time-varying and it cannot be passed through the expectation operator. In Claim 1.1, risk aversion is captured by a constant risk premium $R - R_f > 0$. If risk premiums are time-varying, $\text{cov}_t\,(r^s_{t+1}, -\xi_t)$ is no longer constant and ξ_t plays a role in valuation. In a generalized setting of stochastic interest rates and risk aversion, we must value equity using equation (14) where r_t and ξ_t need to be parameterized.

2. An Affine Information Model

2.1. Discrete-Time Affine Processes

We now specify what are the driving variables in the economy and how they evolve over time. The generalized RIM in equation (14) is able to take into account stochastic short rates and risk aversion. However, equation (14) does not relate realized financial numbers with firm value, since the values on the right hand side of equation (14) are forecasts. In this form, equation (14) gives us a theoretical framework to study risk and return, but not a parametric form to relate firm value with realized financial accounting reports. This section develops a parametric no-arbitrage model of firm value when accounting information is given in dollar amounts.

The LIM of Ohlson (1995) and Feltham and Ohlson (1995) is a parametric formulation of the RIM that expresses the value of a stock as a linear function of current accounting variables specified in dollars, rather than in terms of abstract future expectations. However, it is developed under constant interest rates and risk-neutrality. Our aim is to extend the LIM to account for stochastic interest rates and risk aversion. We specifically ask under what assumptions and parameterizations linearity can survive under more general no-arbitrage conditions. We show it is not always possible to incorporate risk aversion by adjusting a constant discount factor.

For concreteness, we assume the same driving variables in the economy as Feltham and Ohlson (1995). Specifically, suppose the driving variables are denoted by X_t, and let:

$$X_t = \left(x^a_t \; oa_t \; v_{1t} \; v_{2t}\right)'$$

where x^a_t denoting abnormal earnings, oa_t operating assets, v_{1t} and v_{2t} "other" information at time t. We also assume that X_t follows a discrete-time affine process (Duffie and Kan, 1996). (The term "affine" refers to a constant plus linear term.)

Definition 2.1. *A $K \times 1$ vector X_t is said to follow a discrete-time affine process if:*

$$X_{t+1} = \mu + AX_t + \sigma_t \epsilon_{t+1}, \tag{16}$$

where μ is an $K \times 1$ vector, A and σ_t are $K \times K$ matrices and $\epsilon_t \sim N(0,1)$. The conditional *mean* $\mathrm{E}_t(X_{t+1}) = \mu + AX_t$ *is affine in* X_t, *and the conditional covariance is also affine in* X_t *and is given by:*

$$\sigma_t \sigma_t' = h + H \cdot X_t. \tag{17}$$

The notation "·" represents a tensor product, and is interpreted as:

$$H \cdot X_t \equiv \sum_{j=1}^{K} X_{tj} H^{(j)} \tag{18}$$

where X_{tj} *refers to the* j*th element of* X_t. *The* $K \times K$ *matrices* h *and* $H^{(j)}$ *are symmetric.*

Discrete-time affine processes are an attractive parsimonious class of models which can capture feedback (mean-reversion) and stochastic volatility. They capture feedback through the companion matrix A in the conditional mean. Stochastic volatility depends on the level of the variables through the tensor product in the conditional volatility. Hence, the variables in X_t may be heteroskedastic ($H^{(j)} \neq 0$ for some j). Homoskedasticity occurs as a special case when $H^{(j)} = 0 \, \forall j$ and $h \neq 0$.

We give two examples of affine processes. First, if there is no heteroskedasticity in the conditional covariances ($H^{(j)} = 0 \, \forall j$), the process reduces to a Vector Autoregression of first order (VAR(1)). This is the process used in the Feltham-Ohlson LIM, where X_t follows the following VAR(1):

$$X_{t+1} = AX_t + \sigma \epsilon_{t+1} \tag{19}$$

where $X_t = (x_t^a \; oa_t \; v_{1t} \; v_{2t})'$, and $\sigma\sigma' = h$, where h is a constant symmetric matrix. In Feltham and Ohlson (1995) the companion matrix A takes the form:

$$A = \begin{pmatrix} \omega_{11} & \omega_{12} & 1 & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}. \tag{20}$$

Note that the LIM extends to other more general forms of A, rather than to just this special form. This is a special case of the discrete-time affine process, formed by setting $\mu = 0$ and specifying the covariance have no heteroskedasticity, so $H^{(j)} = 0 \, \forall j$.

A second example of an affine process having heteroskedasticity is the Cox, Ingersoll and Ross (CIR) (1985) model of term structure. If $X_t = r_t$, the univariate short rate, then setting $h = 0$ gives a discretized CIR model, where the variance is proportional to the level of the interest rate:

$$X_{t+1} = \mu + AX_t + \sigma_t \epsilon_{t+1}$$

where $\sigma_t^2 = H^{(1)} X_t$, and $H^{(1)}$ is a positive scalar. Under a CIR model, the yield curve can assume a variety of shapes including upward sloping, humped and downward sloping yield curves. The stochastic movements of all interest rates are inferred from the short rate r_t, once the dynamic of r_t in the CIR model is specified. We use the CIR term structure model to incorporate time-varying interest rates.

2.2. An Affine Information Model (AIM)

Under the LIM, firm value is a linear function of accounting information. We state assumptions under which linearity, or an affine form, can be maintained under a more general setting of heteroskedasticity, risk aversion and time-varying interest rates.

For most of this section we assume that spot interest rates are independent of the accounting variables. The interest rate process can be very general. At the end of the section we comment on the case where interest rates and accounting variables are correlated.

2.2.1. An AIM under Independent Interest Rates

We let the K driving variables $X_t = (x_t^a\ v_t')'$, with x_t^a denoting abnormal earnings, and v_t a vector representing "other" information at time t. This formulation subsumes Feltham and Ohlson (1995)'s parameterization, by letting $v_t = (oa_t\ v_{1t}\ v_{2t})'$. We use the affine process of Section 2.1:

Assumption 2.1 *X_t is a $K \times 1$ vector with first element abnormal earnings x_t^a and other elements representing other information at time t. X_t follows a discrete-time affine process as defined by Definition 2.1.*

Next we assume that interest rates follow a process that is consistent with no-arbitrage but independent of accounting information:

Assumption 2.2 *The economy is arbitrage-free, and spot interest rates r_t follow a process independent of X_t. The random variable defined in equation (3) ξ_{t+1} is the product of two factors $\xi_{t+1}^{(r)}$ and $\xi_{t+1}^{(X)}$:*

$$\xi_{t+1} = \xi_{t+1}^{(r)} \cdot \xi_{t+1}^{(X)}, \tag{21}$$

where $\xi_{t+1}^{(r)}$ and $\xi_{t+1}^{(X)}$ are independent and

$$\xi_{t+1}^{(X)} = \exp\left(-\tfrac{1}{2}\gamma'\sigma_t\sigma_t'\gamma + \gamma'\sigma_t\epsilon_{t+1}\right), \tag{22}$$

where ϵ_{t+1} are the shocks of the process of X_t. The only requirement for the factor $\xi_{t+1}^{(r)}$ is that it remains arbitrage-free.

The process for short rates leads to zero coupon bond prices $\Lambda_t^{\{n\}}$ for period n:

$$\Lambda_t^{\{n\}} = \mathrm{E}_t^Q\left[\prod_{j=0}^{n-1}\exp(-r_{t-j})\right]. \tag{23}$$

One example of an admissible process for r_t is a CIR model uncorrelated with X_t, but any term structure uncorrelated with X_t is possible. In the case of the CIR model, zero coupon bond prices are exponential affine functions of r_t:

$$\Lambda_t^{\{n\}} = \exp(a(n) + b(n)r_t),$$

where the $a(n)$ and $b(n)$ coefficients are closed-form (see Cox, Ingersoll and Ross, 1985).

Using the methodology presented in Section 1, an extended version of the Feltham-Ohlson LIM holds, where we extend the LIM to an affine setting.

Proposition 2.1 *Under Assumptions 2.1 and 2.2 the valuation function* $\mathbf{g}_t = S_t - y_t$ *can be expressed as:*

$$\mathbf{g}_t = \alpha_t + \beta_t' X_t, \tag{24}$$

where the constant coefficient α_t *and the linear coefficient* β_t *of the affine form are given by:*

$$\alpha_t = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \Lambda_t^{(n+k)} \right) e_1'(A + \bar{H})^n (\mu + h\gamma)$$

$$\beta_t = \sum_{n=1}^{\infty} \Lambda_t^{\{n\}} e_1'(A + \bar{H})^n, \tag{25}$$

where e_1 *is a* $K \times 1$ *vector with first element 1 and the rest zero,* A *is the companion matrix of the process for* X_t *in equation (16), and* $\bar{H}$ *is a* $K \times K$ *matrix defined as:*

$$\bar{H}_{ij} = \sum_{k=1}^{K} H_{ik}^{(j)} \gamma_k,$$

where $\bar{H}_{ij}$ *is the element in the* i*th row,* j*th column of* $\bar{H}$, $H_{ik}^{(j)}$ *is the element in the* i*th row,* k*th column of* $H^{(j)}$ *(the* $K \times K$ *matrix in equation (18)) and* γ_k *is the* k*th element of* γ.

We make several comments on the affine form of valuation in Proposition 2.1. First, the environment is very general, and Proposition 2.1 shows that an affine form survives with stochastic interest rates, risk aversion and heteroskedasticity in accounting information. However, this affine form in accounting variables X_t depends crucially on the assumption that the interest rate is orthogonal to accounting information. Given this restriction, spot interest rates may take on any dynamic consistent with no-arbitrage. Second, when interest rates are not constant, the affine coefficients α_t and β_t depend on time t through their dependence on the time t zero coupon bond prices $\Lambda_t^{\{n\}}$. In particular, β_t can be interpreted as the valuation of a perpetuity whose payments are risk-adjusted future residual earnings. The discount factors on the perpetuity are $\Lambda_t^{\{n\}}$. Without risk, $\bar{H} = 0$ and $e_1' A^n X_t$ is the future expected abnormal earnings n periods into the future, assuming abnormal earnings have zero mean as in the LIM. Under risk aversion, $\bar{H} \neq 0$, and the future expected abnormal earnings in period $t + n$ incorporate a risk adjustment.

Finally, risk aversion contributes to both α_t and β_t. In the case of homoskedasticity ($H^{(j)} = 0\,\forall j$ and $h \neq 0$), risk aversion gives rise only to state-independent risk premiums in α_t. $\bar{H} = 0$ so the effect of homoskedasticity enters through the action of the $h\gamma$ term. Under heteroskedasticity ($H^{(j)} \neq 0$ for some j), risk aversion enters the β_t terms. In this case, the risk premium is state-dependent, through the non-zero $\bar{H}$ term.

2.2.2. *An AIM Under Constant Interest Rates*

To focus on the effect of risk aversion and heteroskedasticity, we analyze the case where interest rates are constant ($r_t = r_f\,\forall t$, $R_f = \exp(r_f)$ and $\Lambda_t^{\{n\}} = R_f^{-n}$). In this case, the valuation formula in Proposition 2.1 can be further simplified because all bond pricies are geometrically related. We look at several examples, including the original LIM. We also determine when linearity can be maintained under risk aversion and heteroskedasticity.

Corollary 2.1 *In the case of constant interest rates,* $\Lambda_t^{\{n\}} = R_f^{-n}$*, the coefficients are constant:* $\alpha_t = \alpha$ *and* $\beta_t = \beta$ *and are given by:*

$$\alpha = \frac{R_f}{R_f - 1} e_1' (R_f - A)^{-1}(\mu + h\gamma)$$
$$\beta = (R_f - (A + \bar{H})')^{-1}(A + \bar{H})' e_1. \tag{26}$$

Note the constant term α can arise through either a non-zero μ, or as a constant risk premium through homoskedastic risk ($h \neq 0$).

Previously, applications of the RIM account for risk by using an *ad hoc* adjustment to a constant discount factor. Corollary 2.1 shows that with constant interest rates, it may not be possible to incorporate the effects of risk aversion in this way. The traditional RIM model in Claim 1.1 incorporates risk aversion by setting the constant discount rate R to be $R = R_f + \bar{\sigma}$, where the spread $\bar{\sigma}$ over the risk-free rate takes risk aversion into account:

$$\mathbf{g}_t^{\text{cdf}} = \sum_{i=1}^{\infty} R^{-i} \mathrm{E}_t \left[x_{t+i}^a \right], \tag{27}$$

where "cdf" denotes constant discount factor. The following lemma shows that under a constant discount rate R, the valuation function $\mathbf{g}_t^{\text{cdf}}$ is linear in X_t and does not have a constant term:

Lemma 2.1 *Under Assumption 2.1 and* $\mu = 0$*,* $\mathbf{g}_t^{cdf}$ *is a linear function of* X_t*.*

Corollary 2.1 shows that the constant term is zero only if $\mu = 0$ and $\gamma = 0$, or $\mu = 0$ and $h = 0$. If either condition is not satisfied, the value function $\mathbf{g}_t$ cannot be described as in equation (27) using a constant discount factor R, so we have the following corollary:

Corollary 2.2 *It is not always possible, even under constant interest rates, to have a constant discount factor.*

Note that Feltham and Ohlson (1995) parameterize x_t^a to have zero mean, but if some state variables in X_t have non-zero mean then α is no longer zero. In this case, only the assumption of risk neutrality guarantees the absence of a constant risk premium.

The following example shows how the AIM nests the LIM of Ohlson (1995) and Feltham and Ohlson (1995) as a special case.

Example 2.1 ***The Feltham-Ohlson LIM.*** *In the case of $\mu = 0$, homoskedasticity ($H^{(j)} = 0$ so $\bar{H} = 0$) and risk neutrality, $\gamma = 0$, the AIM reduces to the Feltham-Ohlson LIM, where:*

$$\alpha = 0$$
$$\beta = (R_f - A')^{-1} A' e_1.$$

We comment that under homoskedasticity, if $\mu \neq 0$ then α is no longer zero but the linear coefficient β in the traditional LIM remains unchanged.

In the next example we state an alternative set of conditions under which linearity can be maintained under risk aversion.

Example 2.2 *In the case $\mu = 0$, $h = 0$ and risk aversion ($\gamma \neq 0$), the valuation function g_t has a linear form, where:*

$$\alpha = 0$$
$$\beta = (R_f - (A + \bar{H})')^{-1} (A + \bar{H})' e_1.$$

Since we assume no homoskedastic risk ($h = 0$), X_t must exhibit heteroskedasticity ($H^{(j)} \neq 0$ for some j) to be non-degenerate. In this case, the effect of risk aversion is absorbed into the linear coefficient β (through the $\bar{H}$ term). That is, state-dependent risk (through heteroskedasticity and risk aversion) gives rise to state-dependent risk premiums.

2.2.3. The Case of Correlated Interest Rates

When interest rates are correlated with the accounting variables, they must be included as a state variable in X_t. That is, we re-define X_t as $X_t = (r_t\, x_t^a\, v_t')'$. Now, the random variable ξ_{t+1} can no longer be factored into two independent terms, one depending on r_t and one depending only on accounting variables. In this case, an affine solution for $\mathbf{g}_t$ is no longer possible, but we can still find a functional form to relate $\mathbf{g}_t$ with contemporaneous X_t. It can be shown that:

$$\mathbf{g}_t = \sum_{i=1}^{\infty} e^{a(i)+b(i)'X_t} (c(i) + d(i)'X_t),$$

where $a(i)$ and $c(i)$ are scalars, and $b(i)$ and $d(i)$ are vectors. The coefficients $a(i)$, $b(i)$, $c(i)$ and $d(i)$ are constant and can be derived similarly to Proposition 3.1.[10] This formula still relates $\mathbf{g}_t$ to observed values of X_t but the relation is now non-linear.

3. A Parametric Generalized Earnings Model

While the AIM or LIM presents firm value as a linear function of earnings, it is not convenient for determining how growth in earnings or growth of the firm affects valuation. This section uses a measure of firm profitability and growth in book, together with the stochastic short rate, as driving factors to build a model of the price-to-book ratio. This allows direct inference of how the rate of profitability and firm growth affect ratio valuation. We introduce the framework in Section 3.1. Section 3.2 presents an example of a specialized process for the driving variables to motivate how the model can be used in comparative statics. We derive the model in Section 3.3. Finally, Section 3.3 conducts comparative statics using the price-to-book formula.

3.1. Normalizing the RIM by Book Value

We start by repeating the no-arbitrage RIM in equation (14):

$$S_t = y_t + \sum_{i=1}^{\infty} \mathrm{E}_t^Q \left[\left(\prod_{j=0}^{i-1} \exp(-r_{t+j}) \right) x_{t+i}^a \right].$$

To deal with accounting ratios, we normalize and divide each side by book value y_t:

$$\frac{S_t}{y_t} = 1 + \mathrm{E}_t^Q \left[\sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \exp(-r_{t+j}) \right) \left(\frac{x_{t+i}}{y_{t+i-1}} - (e^{r_{t+i-1}} - 1) \right) \frac{y_{t+i-1}}{y_t} \right] \tag{28}$$

We introduce some definitions to convert the accounting variables in levels to ratios or growth rates:

$$\begin{aligned} B_t &= S_t / y_t \\ e^{g_t} &= y_t / y_{t-1} \\ r_t^e &= x_t / y_{t-1} \\ z_t &= r_t^e - (\exp(r_{t-1}) - 1) \end{aligned} \tag{29}$$

In equation (29), B_t is the price-to-book ratio and g_t is the growth rate in book value. The next equation for r_t^e is the accounting return on equity (earnings to book). Higher accounting returns denote higher profitability. The variable z_t is the accounting return in excess of the risk-free rate, which we define as "the abnormal accounting return." It is derived using the Feltham and Ohlson (1999) definition of abnormal earnings, and then normalizing the abnormal earnings by book value:

$$\begin{aligned} x_t^a &= x_t - (\exp(r_{t-1}) - 1) y_{t-1} \\ z_t &\equiv \frac{x_t^a}{y_{t-1}} = \frac{x_t}{y_{t-1}} - (\exp(r_{t-1}) - 1) \frac{y_{t-1}}{y_{t-1}} \end{aligned} \tag{30}$$

Under mark-to-market accounting $B_t = 1$ and $z_t = 0$. Under conservative accounting z_t may be non-zero.

Using the above definitions we can rewrite equation (28) as:

$$\begin{aligned} B_t &= 1 + e^{-r_t} \mathrm{E}_t^Q \left[z_{t+1} + \sum_{i=2}^{\infty} \left(\prod_{j=1}^{i-1} e^{-(r_{t+j} - g_{t+j})} \right) z_{t+i} \right] \\ &= 1 + e^{-r_t} \mathrm{E}_t^Q \left[\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i-1} e^{-(r_{t+j} - g_{t+j})} \right) z_{t+i} \right] \end{aligned} \tag{31}$$

where we assume that the product term is equal to 1 if the index is negative. We assume that the variables B_t, r_t, z_t and g_t are stationary.[11]

Equation (31) rewrites the RIM, but expresses the price-to-book ratio as discounted abnormal returns. It still remains in the no-arbitrage setting of the RIM. We remark that the normalized setting of equation (31) is still Miller-Modigliani (1961) consistent, so dividend policy does not influence value because the original RIM in levels is Miller-Modigliani consistent (see Ohlson, 1995). The driving variables behind the price-to-book are accounting returns of earnings (rather than earnings for price levels), growth in book value (rather than book value in levels), and accounting abnormal returns of earnings (rather than abnormal earnings). This moves us from the setting of dollar amounts or price levels to ratios or growth rates.

Although equation (31) shows the price-to-book ratio to be a function of spot rates, abnormal returns and growth, the numbers on the right hand side of equation (31) are forecasts. We need a parameterization of the driving variables to relate the price-to-book to contemporaneous accounting information. We do in this in the following section.

3.2. A Specialized Process for Interest Rates, Profitability and Growth

We specify the driving variables of the economy as the risk-free rate, abnormal returns of earnings and growth in book value. Denote $X_t = (r_t\, z_t\, g_t)'$ and assume that X_t follows a discrete-time affine process, as defined in equations (16) and (17). To give a concrete example, suppose X_t is given by:

$$\begin{aligned} r_{t+1} &= \mu_r + \rho_r r_t + \sigma_r \sqrt{r_t} \epsilon_{t+1}^1 \\ z_{t+1} &= \mu_z + \alpha_1 r_t + \alpha_2 z_t + \sigma_z \epsilon_{t+1}^2 \\ g_{t+1} &= \mu_g + \alpha_3 r_t + \alpha_4 z_t + \alpha_5 g_t + \sigma_g \epsilon_{t+1}^3 \end{aligned} \tag{32}$$

with $\epsilon_t = (\epsilon_t^1\, \epsilon_t^2\, \epsilon_t^3)' \sim N(0,\mathrm{I})$.

This structure may seem overly restrictive, but it is only intended as a simple example of how feedback dynamics can be accomplished in the affine system. Assuming the errors are independent implies that interest rates, abnormal earnings and growth in equity are subject to independent shocks. We make this assumption so that we can analyze the effect of each of the variables in X_t separately. The first equation is a discretized square root process (the workhorse CIR model of term structure asset pricing) of the short rate. Through the variance term, conditional volatility increases proportionally with the level of the interest rate.

The second equation is a Gaussian process which says that abnormal earnings are autocorrelated, and that the short rate Granger-causes abnormal returns. As interest rates go up, we expect abnormal earnings to decrease (see Nissim and Penman, 2000). This is captured by a negative α_1 coefficient. Increasing interest rates decrease the discount factors applying in future periods and decrease the price-to-book through a "denominator" effect. The predictability of accounting returns by interest rates is a "numerator" effect because it decreases cashflows in future periods. Both the denominator and numerator effect are handled simultaneously by the dynamics of companion matrix A in the discrete-time affine process (equation (16)). The coefficient α_2 captures the mean-reversion of profitability. As profitability becomes more mean-reverting (or less persistent), α_2 decreases.

The last equation parameterizes growth as a Gaussian process. The conditional mean of growth in equity is a predictable function of past growth in equity, lagged short rates and abnormal earnings. In particular, we would expect firm growth and profitability to be positively related; this would be captured by a positive α_4 coefficient. The α_5 coefficient reflects mean-reversion or persistence of growth in book. As mean-reversion increases (or persistent decreases), α_5 decreases.

In terms of the notation of Section 1.1, this imposes the following structure on the companion matrix A, and the matrices h and H driving the covariances:

$$A = \begin{pmatrix} \rho_r & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & \alpha_5 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_z^2 & 0 \\ 0 & 0 & \sigma_g^2 \end{pmatrix}, \qquad H^{(1)} = \begin{pmatrix} \sigma_r^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $H^{(2)} = H^{(3)} = 0$.

We examine several interesting effects on the price-to-book ratio from this parameterization. In particular, we separately determine the effect of changing the parameters of the short rate, abnormal return or growth in book on the price-to-book. Changing the mean reversion of profitability or growth is accomplished by changing α_2 and α_5. Risk-averse agents are affected by changes in volatility, so σ_r, σ_z and σ_g affect valuation under risk aversion. To conduct comparative statics, however, we need to derive an analytical expression for the price-to-book ratio.

3.3. The Parametric Earnings Model

We now develop a non-linear formula for the price-to-book ratio. The case presented in the previous section is only an example of a particular affine system, but our derivations presented here apply to the most general affine model. This formula is like the LIM in that it relates the price-to-book to observed accounting data rather than to forecasts in equation (31), but with abnormal returns and growth rates a linear solution is no longer possible. The price-to-book valuation formula is given in the following proposition, which evaluates the conditional expectation of the infinite sum in equation (31) as a function only of time t information.

Proposition 3.1 *Suppose the variables $X_t = (r_t\, z_t\, g_t)'$ follow a discrete-affine process in Definition 2.1 and the default-adjusted pricing kernel π_{t+1} takes the log-linear form in equation (5). Then the price-to-book ratio can be written only as a function of time t information:*

$$B_t = 1 + e^{-r_t} \mathrm{E}_t^Q \left[\sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} e^{-(r_{t+i} - g_{t+i})} \right) z_{t+i} \right]$$
$$= 1 + e^{-r_t} \sum_{i=1}^{\infty} e^{a(i)+b(i)'X_t} (c(i) + d(i)'X_t), \tag{33}$$

where $a(i)$ and $c(i)$ are scalars, and $b(i)$ and $d(i)$ are 3×1 vectors. The constant coefficients $a(i)$, $b(i)$, $c(i)$ and $d(i)$ are given by:

$$\begin{aligned} a(i) &= a(i-1) + (-e_1 + e_3 + b(i-1))'(\mu + h\gamma) \\ &\quad + \tfrac{1}{2}(-e_1 + e_3 + b(i-1))'h(-e_1 + e_3 + b(i-1)) \\ b(i)'X_t &= (-e_1 + e_3 + b(i-1))'(AX_t + H \cdot X_t\gamma) \\ &\quad + \tfrac{1}{2}(-e_1 + e_3 + b(i-1)'(H \cdot X_t)(-e_1 + e_3 + b(i-1)) \\ c(i) &= c(i-1) + d(i-1)'(\mu + h(\gamma - e_1 + e_3 + b(i-1))) \\ d(i)'X_t &= d(i-1)'(AX_t + (H \cdot X_t)(\gamma - e_1 + e_3 + b(i-1))), \end{aligned} \tag{34}$$

where e_j denotes a vector of zeros with a 1 in the jth place. The initial conditions are given by:

$$\begin{aligned} a(1) &= 0 \\ b(1)'X_t &= 0 \\ c(1) &= e_2'(\mu + h\gamma) \\ d(1)'X_t &= e_2'((H \cdot X_t)\gamma + AX_t) \end{aligned} \tag{35}$$

Let us interpret the valuation model of Proposition 3.1 by comparing it to the LIM and the AIM in Section 2. Although the LIM is given in dollar amounts and Proposition 3.1 gives a model in ratio terms the two approaches are similar. First, the LIM relates how prices are related to current observable accounting information rather than to forecasts. Proposition 3.1 does the same thing. It assumes a parameterization of accounting ratios and growth rates ($X_t = (r_t\, z_t\, g_t)'$) that enables the infinite sum involving expectations in equation (31) to be a function only of observable X_t information. Second, the AIM presented in Section 2 incorporates time-varying interest rates and risk aversion under heteroskedasticity. There, to maintain an affine form r_t must be uncorrelated with accounting variables. Here, r_t is correlated with accounting ratios and included as a state variable. Third, both the AIM and Proposition 3.1 are derived under no-arbitrage framework. Finally, the LIM is closed-form and gives price as a linear function of accounting information. Proposition 3.1 is also closed-form, but the solution is non-linear.

We interpret the coefficients $a(i)$, $b(i)$, $c(i)$ and $d(i)$ in the closed-form solution as follows. As long as transversality is satisfied, $a(i) \to -\infty$ when $i \to \infty$ so the exponential tends to zero, and the individual terms in the sum quickly become small. Practically, this means that the sum in equation (33) can be evaluated very quickly without many terms.

The $a(i)$ and $b(i)$ coefficients result from the effect of the $r-g$ discounting terms. In equation (34), the terms quadratic in $-e_1+e_3+b(i-1)$ are Jensen's inequality terms. The terms linear in $-e_1+e_3+b(i-1)$ involving γ result from risk aversion. The Jensen's inequality terms are always positive, while the risk premium terms can be negative if γ is negative. Increasing the volatility of a factor increases the Jensen's inequality terms, unless the risk premium terms in the $a(i)$ and $b(i)$ recursions outweigh the effect of the Jensen's inequality terms. This implies that in a risk-neutral setting, increasing the volatility of a factor increases, *ceteris paribus,* the terms in the exponential. This increases the price-to-book. However, in a risk-averse setting, that is when $\gamma < 0$, the price-to-book can decrease with volatility. We illustrate this in the next section.

The $c(i)$ and $d(i)$ coefficients value the abnormal return stream. Notice from the initial conditions in equation (35) that the e_2 vector pulls out only the abnormal returns terms. The terms involving μ and AX_t result from the action of the conditional mean of the process of X_t, and the terms involving γ are the risk premiums which act on the covariances. In the recursions for $c(i)$ and $d(i)$ in equation (34) the Jensen term $-e_1+e_3+b(i-1)$ also enters, along with a risk premium effect.

Equations (34) and (35) completely determine the response of the price-to-book (equation (33)) in terms of parameters to the underlying process X_t. However, the reaction of the price to changes in the parameters is not immediately transparent due to the recursive nature of the coefficients in the valuation equation. We now conduct a series of exercises in comparative statics to further analyze the effects of parameter changes on the price-to-book.

3.4. Comparative Statics of the Price-to-Book

In this section we show how varying the parameters of the processes of the short rate, abnormal returns and growth in book affect the price-to-book ratio of a firm, using the motivating example in equation (32). In a previous version of this paper (Ang and Liu, 1998), we calibrated this model to several individual stocks. We use the estimated parameters of Intel, from January 1975 to June 1997, as a basis for illustrating the comparative statics. These parameters are listed in Table 1. We conduct our comparative statics at the base case of the sample mean for X_t over the sample.[12] In our plots, we show this baseline case as a circle.

In our comparative static exercises we wish to clarify the role of risk aversion. To do this, only one factor, the growth in equity g_t, is priced, and we set the prices of risk for the short rate r_t and abnormal returns z_t to zero. The first assumption is close to reality, because term structure estimations have found insignificant prices of interest rate risk. The action of the price of risk of z_t is very similar to the price of risk of g_t, so we concentrate only on the action of one price of risk. Hence we set $\gamma = (0\,0\,\gamma_3)'$ where γ_3 is the price of risk of growth in book.

Table 1. Parameters for comparative statics

Baseline Parameters	
μ_r	0.0054
ρ_r	0.7548
σ_r	0.0374
μ_z	0.0263
α_1	−0.8695
α_2	0.7720
σ_z	0.0176
μ_g	0.0247
α_3	1.1829
α_4	0.6008
α_5	−0.0105
σ_g	0.0698
γ_3	−13.1982

Notes: These parameters are the base-line case for the comparative statics exercises. The equations for the processes are given by equation (32). We set $\gamma_1 = \gamma_2 = 0$.

For reference, we repeat the system $X_t = (r_t\ z_t\ g_t)'$ here:

$$\begin{aligned} r_{t+1} &= \mu_r + \rho_r r_t + \sigma_r \sqrt{r_t}\epsilon^1_{t+1} \\ z_{t+1} &= \mu_z + \alpha_1 r_t + \alpha_2 z_t + \sigma_z \epsilon^2_{t+1} \\ g_{t+1} &= \mu_g + \alpha_3 r_t + \alpha_4 z_t + \alpha_5 g_t + \sigma_g \epsilon^3_{t+1}. \end{aligned} \tag{36}$$

3.4.1. *Effect of the Short Rate on the Price-to-Book*

We first examine the effect of the short-rate parameters. As μ_r increases, short rate levels increase and future abnormal earnings are discounted back at higher rates. This decreases the price-to-book. In Figure 1 we see the price-to-book as a function of the persistence ρ_r, and the volatility σ_r. We first discuss the effect of ρ_r. As interest rates become more persistent, the price-to-book decreases. Intuitively, increasing the persistence increases the unconditional mean of the short rate. As cash flows are generally valued back at a higher discount factor, the price-to-book falls.

In Figure 1 the price-to-book increases when σ_r increases. At first glance, this would seem counter-intuitive, for two reasons. First, we would expect agents to dislike volatility, so that price-to-book would decrease when volatility increases. Second, the interest rate is bounded at zero in the CIR model, and increasing the volatility of the short rate increases the unconditional mean of interest rates. This implies that the discount factors which value back future abnormal returns are higher. However, in our base-line case, agents are risk-neutral with respect to the interest rate risk. The increase in the price-to-book when σ_r increases is purely due to a Jensen's inequality effect. Only if interest rate risk is priced is it possible to cancel the Jensen's inequality effect.

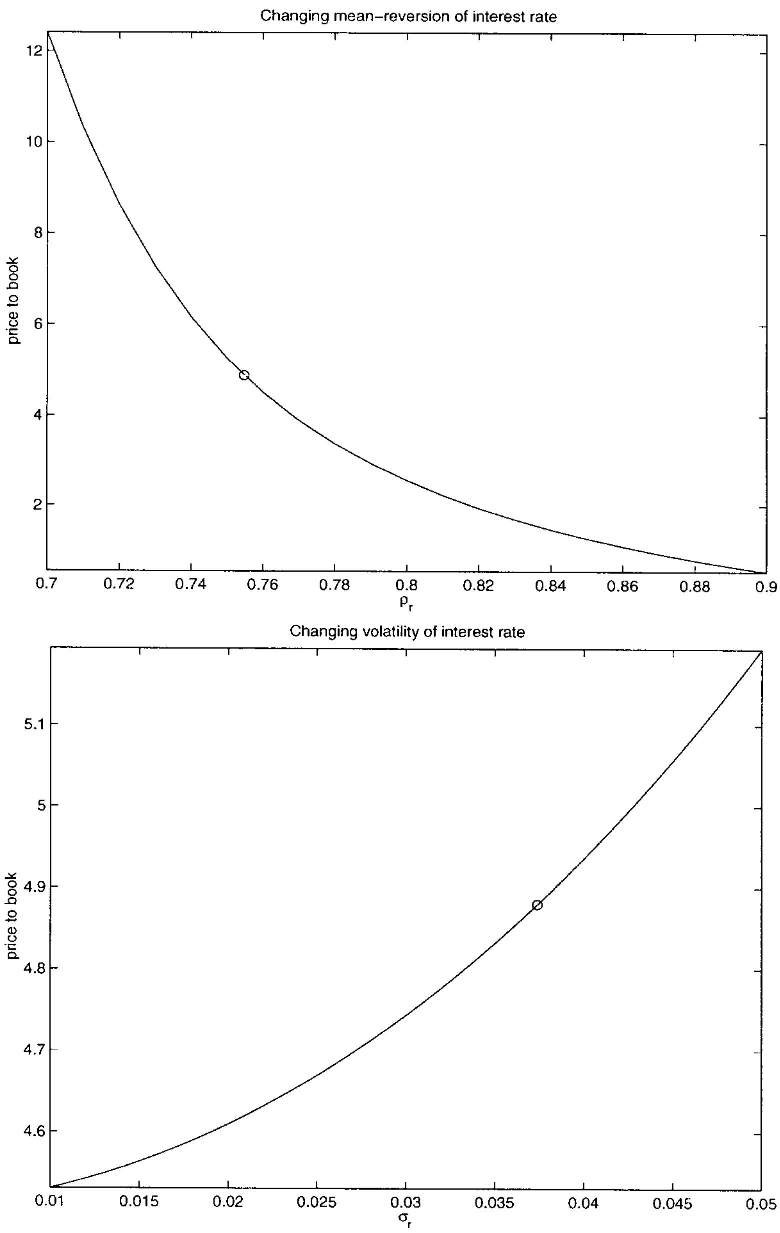

Figure 1. Comparative statics for short rate parameters. (The figure plots the price-to-book as a function of persistence of the short rate ρ_r (top plot), and as a function of the short-rate volatility σ_r (bottom plot). The base-line case is shown as a circle.)

3.4.2. Effect of Profitability on the Price-to-Book

We turn now to the effect of profitability on the price-to-book. As expected, increasing μ_z increases the price-to-book as a higher μ_z implies greater profitability. The price-to-book ratio also increases as the predictability coefficient α_1 increases, if z_t is positive. Economically, this occurs because increasing abnormal returns causes cashflows in future periods to increase, and hence this increases the price-to-book ratio. Correspondingly, decreasing α_1 decreases abnormal returns. The effect is opposite for negative levels of z_t.

The top panel of Figure 2 shows the effect of altering the persistence (α_2) of abnormal returns. Increasing the persistence of z_t, or decreasing the mean-reversion, increases the price-to-book. The statistical interpretation is as follows. The unconditional mean of abnormal returns rises as the persistence rises. Higher average abnormal returns then imply higher price-to-book. Economically, we expect a firm's profitability to mean-revert within and across industries (see Fama and French, 2000). Higher mean reversion (or lower persistence of abnormal earnings) means that high relative earnings in the short term persist for fewer periods. This lower profitability decreases the price-to-book ratio.

The bottom panel of Figure 2 presents the price-to-book as a function of abnormal return volatility σ_z. In our parameterization, increasing the volatility of abnormal returns increases the price-to-book. This is because the price of risk of abnormal returns γ_2 is zero, so agents are risk-neutral with respect to profitability. The Jensen's effect causes the price-to-book to increase when σ_z increases. However, if γ_2 is negative and agents are risk-averse with respect to abnormal returns, then it may be possible for the price-to-book to fall as volatility of profitability increases.

3.4.3. Effect of Growth on the Price-to-Book

Any parameters which increase the growth in book increase the price-to-book ratio. For example, increasing μ_g, α_3 or α_4 when z_t is positive increases the price-to-book because each parameter raises growth in book. The intuition is that increasing growth increases the likelihood of higher cashflows in future periods.

The parameter α_5 captures the persistence, or mean-reversion, of firm growth. The top panel of Figure 3 shows that increasing the persistence of the growth in book increases the price-to-book. This effect is similar to increasing the persistence of z_t (α_2), since the unconditional mean of g_t rises as the persistence of g_t increases. Higher persistence implies that firm growth mean-reverts to an industry or market average at a faster rate. Hence the firm has fewer periods to enjoy the benefits of relatively higher growth.

The bottom panel of Figure 3 shows the effect of a decreasing price-to-book with increasing volatility in growth in equity (σ_g). This is in line with intuition: we would expect, *ceteris paribus,* normal risk-averse investors to lower their valuations the greater the volatility in growth. We obtain this result because there is a large non-zero price of risk on growth in equity γ_3 and this causes the price-to-book to decrease as volatility increases. In this case, in addition to the Jensen's inequality effect (which increases with σ_g), there is also a risk aversion effect which counteracts the Jensen's inequality effect.

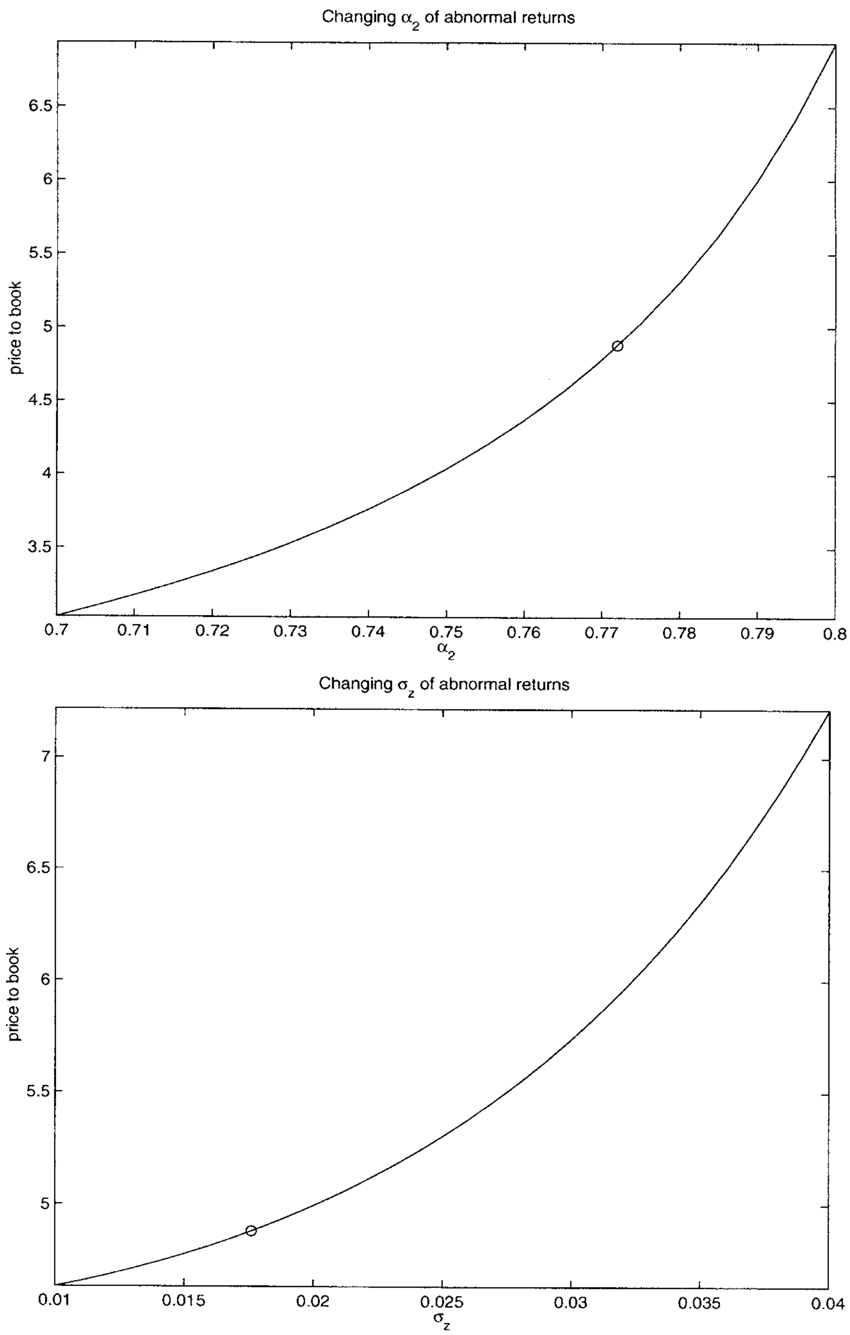

Figure 2. Comparative statics for abnormal return parameters. (The figure plots the price-to-book as a function of the persistence of abnormal returns α_2 (top plot), and the volatility of abnormal returns σ_z (bottom plot). The base-line case is shown as a circle.)

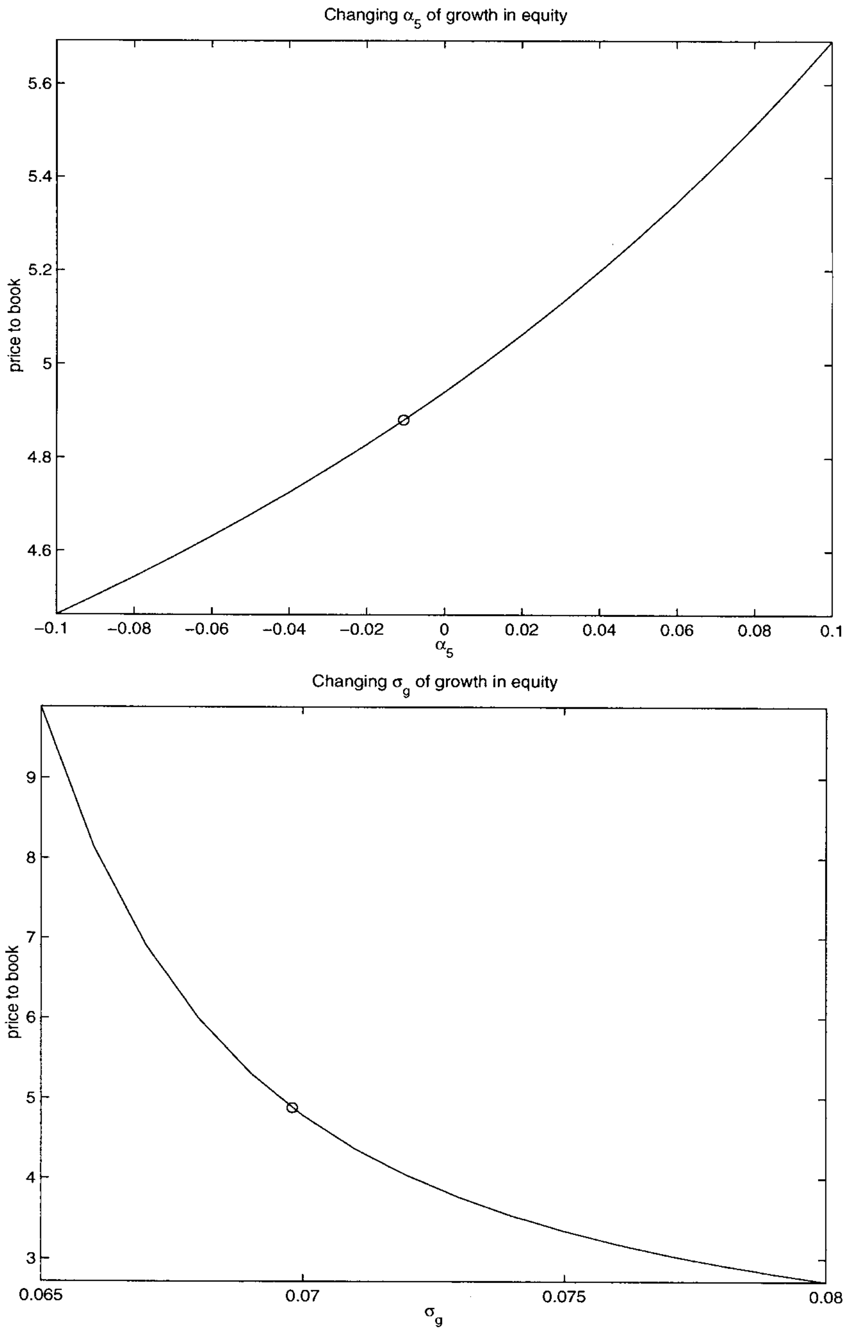

Figure 3. Comparative statics for growth in equity parameters. (The figure plots the price-to-book as a function of the persistence of growth in equity α_5 (top plot), and the volatility of growth in equity σ_g (bottom plot). The base-line case is shown as a circle.)

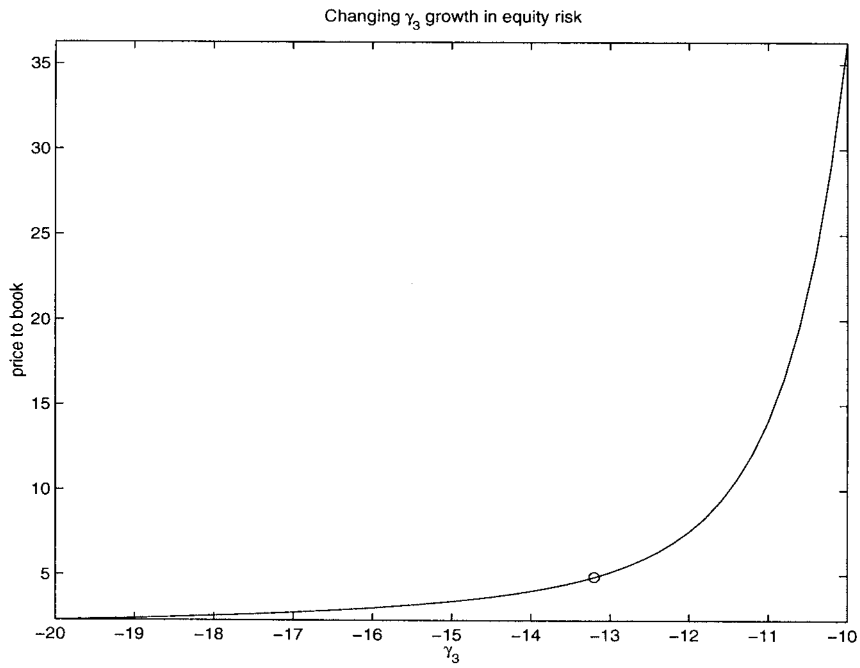

Figure 4. Comparative statics for price of risk of growth in equity. (The figure plots the price-to-book as a function of the price of risk of growth in equity (γ_3). The base-line case is shown as a circle.)

Finally, Figure 4 shows the effect of risk aversion on the price-to-book. In Figure 4, for risk aversion levels below $\gamma_3 = -15$ the price-to-book is very flat, but as investors approach risk neutrality the price-to-book becomes very large.

4. Conclusion

This paper introduces a methodology that incorporates stochastic interest rates, risk aversion and heteroskedasticity into the Residual Income Model (RIM). We provide two applications of the methodology. First, in applying the methodology to dollar amounts, we show that the Ohlson (1995) and Feltham and Ohlson (1995) Linear Information Model generalizes to an affine (constant plus linear terms) model under time-varying interest rates and risk aversion. The processes for accounting information may be heteroskedastic. The interest rate process is very general but is assumed to be uncorrelated with the processes governing the evolution of accounting information.

Second, in applying the methodology to ratio dynamics, we provide a non-linear closed-form formula for the price-to-book ratio in terms of stochastic short rates, profitability and firm growth. In comparative static exercises, increasing the growth in book increases the price-to-book ratio and increasing the mean-reversion of profitability or growth decreases the price-to-book. The effect of interest rates on the price-to-book depends on the degree of risk aversion. Under sufficiently high risk aversion, increasing the mean or volatility of the short rate decreases the price-to-book ratio.

Appendix Proofs

We start by stating the following Lemma which gives the expectation of the product of a normal with the exponential of a normal, which can be proved by evaluating the expectation. This is used in some of the proofs below.

A. Lemma

Lemma A.1 *If Y is distributed as a K-variate normal with $Y \sim N(0, \Sigma)$, and γ and δ are $K \times 1$ constant vectors then*

$$E(\delta' Y e^{\gamma' Y}) = \delta' \Sigma \gamma e^{\frac{1}{2}\gamma' \Sigma \gamma} \tag{A.1}$$

B. Proof of Claim 1.1

Starting from the relation $S_t = \mathrm{E}_t[\pi_{t+1}(S_{t+1} + \delta_{t+1})]$ we can substitute for $\pi_{t+1} = R_f^{-1}\xi_{t+1}$ using the assumptions about a flat term structure to get:

$$S_t = R_f^{-1}\mathrm{E}_t(\xi_{t+1}(S_{t+1} + \delta_{t+1})). \tag{B.1}$$

Using the definition of the return of the stock $r^s_{t+1} \equiv (S_{t+1} + \delta_{t+1})/S_t$ and $\mathrm{cov}_t(r^s_{t+1}, -\xi_{t+1}) = \bar{\sigma}$, we can write $\mathrm{E}_t(r^s_{t+1}) = R_f + \bar{\sigma} \equiv R$, with R a constant.

In this case:

$$S_t = \frac{1}{R}\mathrm{E}_t(S_{t+1} + \delta_{t+1}), \tag{B.2}$$

and iterating this equation forward and assuming transversality we obtain:

$$S_t = \sum_{i=1}^{\infty} R^{-i}\mathrm{E}_t(\delta_{t+i}). \tag{B.3}$$

Then, as in Ohlson (1995), abnormal earnings become $x^a_t = x_t - (R-1)y_{t-1}$, and substituting for dividends in the previous equation yields:

$$S_t = y_t + \mathrm{E}_t\left[\sum_{i=1}^{\infty} R^{-i}x^a_{t+i}\right] = y_t + \sum_{i=1}^{\infty} R^{-i}\mathrm{E}_t\left(x^a_{t+i}\right) \tag{B.4}$$

because the telescoping sum collapses, assuming $R^{-\tau}\mathrm{E}_t(y_{t+\tau}) \to 0$ as $\tau \to \infty$.

C. Proof of Proposition 2.1

From equation (14) we can write:

$$\mathbf{g}_t = \sum_{i=1}^{\infty} \mathrm{E}_t^Q \left[\left(\prod_{j=0}^{i-1} \exp(-r_{t+j}) \right) x_{t+i}^a \right] \tag{C.1}$$

where $\mathbf{g}_t = S_t - y_t$. An equivalent statement is:

$$\begin{aligned} \mathbf{g}_t &= \mathrm{E}_t\big[\exp(-r_t)\xi_{t+1}\big(\mathbf{g}_{t+1} + x_{t+1}^a\big)\big] \\ \exp(r_t)\mathbf{g}_t &= \mathrm{E}_t\big(\mathbf{g}_{t+1} + x_{t+1}^a\big) + \mathrm{cov}_t\big(\mathbf{g}_{t+1} + x_{t+1}^a, \xi_{t+1}\big), \end{aligned} \tag{C.2}$$

noting that $\mathrm{E}_t(\xi_{t+1}) = 1$.

Next, conjecture an affine solution for $\mathbf{g}_t$ which has the form:

$$\mathbf{g}_t = \alpha_t + \beta_t X_t,$$

where α_t and β_t can depend on zero coupon bond prices $\Lambda_t^{\{n\}}$, but not the state variables X_t. We can rewrite equation (C.2) as:

$$\big(\Lambda_t^{\{1\}}\big)^{-1}(\alpha_t + \beta_t' X_t) = \mathrm{E}_t(\alpha_{t+1} + (\beta_{t+1} + e_1)' X_{t+1})\mathrm{cov}_t((\beta_{t+1} + e_1)'\sigma_t \epsilon_{t+1}, \xi_{t+1}). \tag{C.3}$$

Here note that $\exp(r_t) = (\Lambda_t^{\{1\}})^{-1}$. Using Lemma A.1 and evaluating the conditional mean of X_{t+1} gives:

$$\begin{aligned} \big(\Lambda_t^{\{1\}}\big)^{-1}\alpha_t + \big(\Lambda_t^{\{1\}}\big)^{-1}\beta_t' X_t = \mathrm{E}_t(\alpha_{t+1}) &+ (\mathrm{E}_t(\beta_{t+1}) + e_1)'\mu + (\mathrm{E}_t(\beta_{t+1}) + e_1)' A X_t \\ &+ (\mathrm{E}_t(\beta_{t+1}) + e_1)'(h + H \cdot X_t)\gamma, \end{aligned} \tag{C.4}$$

after substituting $\sigma_t \sigma_t' = h + H \cdot X_t$. Equating the coefficients of X_t we have:

$$\begin{aligned} \big(\Lambda_t^{\{1\}}\big)^{-1}\alpha_t &= \mathrm{E}_t(\alpha_{t+1}) + (\mathrm{E}_t(\beta_{t+1}) + e_1)'(\mu + h\gamma) \\ \big(\Lambda_t^{\{1\}}\big)^{-1}\beta_t &= (A + \bar{H})'(\mathrm{E}_t(\beta_{t+1}) + e_1). \end{aligned} \tag{C.5}$$

One can show by substitution that the above equations are solved by setting:

$$\begin{aligned} \alpha_t &= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \Lambda_t^{(n+k)} \right) e_1'(A + \bar{H})^n(\mu + h\gamma) \\ \beta_t &= \sum_{n=1}^{\infty} \Lambda_t^{\{n\}} e_1'(A + \bar{H})^n. \end{aligned} \tag{C.6}$$

D. Proof of Lemma 2.1

Since R is a constant, $x_t^a = x_t - (R-1)y_{t-1}$ is a linear function of the state variables X_t. That is, $x_t^a = L' X_t$ for some constant vector L. Then, $\mathrm{E}_t(x_{t+i}^a) = L' A^i X_t$ is a linear function of X_t. Hence:

$$\mathbf{g}_t = \sum_{i=1}^{\infty} R^{-i} L' A^i X_t = L' A (R - A)^{-1} X_t, \tag{D.1}$$

which is a linear function of X_t, that is, there is no constant term.

E. Proof of Proposition 3.1

To prove Proposition 3.1 we show that a single term in the infinite sum at horizon T sum takes the form:

$$\mathrm{E}_t^Q\left[\left(\prod_{i=1}^{T-1} e^{-(r_{t+i}-g_{t+i})}\right) z_{t+T}\right] = e^{a(T)+b(T)'X(t)}(c(T) + d(T)'X(t)) \tag{E.1}$$

We show that the coefficients $a(T)$, $b(T)$, $c(T)$, and $d(T)$ are given by the Ricatti difference equations in equation (34) with initial conditions in equation (35). Once this is shown, Proposition 3.1 follows immediately from evaluating each individual term in the infinite sum.

We prove equation (E.1) by induction. Assume validity of equation (E.1) for T. We show that the equation holds for $T + 1$. Using iterative expectations we can write:

$$\begin{aligned}\mathrm{E}_t^Q\left[\left(\prod_{i=1}^{T} e^{-(r_{t+i}-g_{t+i})}\right) z_{t+T+1}\right] &= \mathrm{E}_t^Q\left[e^{-(r_{t+1}-g_{t+1})}\mathrm{E}_{t+1}^Q\left[\left(\prod_{i=1}^{T-1} e^{-(r_{t+1+i}-g_{t+1+i})}\right) z_{t+T+1}\right]\right] \\ &= \mathrm{E}_t^Q\left[e^{-(e_1-e_3)'X_{t+1}} e^{a(T)+b(T)'X_{t+1}}(c(T) + d(T)'X_{t+1})\right].\end{aligned} \tag{E.2}$$

The induction assumption is used in the last equality. We then observe that X_{t+1} under Q satisfies (this is the discrete-time version of Girsanov's theorem):

$$X_{t+1} = \mu + AX_t + \sigma_t\sigma_t'\gamma + \sigma_t\epsilon_{t+1}^Q, \tag{E.3}$$

where ϵ_{t+1}^Q is a (mean-zero) standard normal random variable under Q. Then:

$$\begin{aligned}&\mathrm{E}_t^Q\left[\left(\prod_{i=1}^{T} e^{-(r_{t+i}-g_{t+i})}\right) z_{t+T+1}\right] \\ &\quad = e^{a(T)+(-e_1+e_3+b(T))'(\mu+AX_t+\sigma_t\sigma_t'\gamma)} \\ &\qquad \times \mathrm{E}_t^Q\left[e^{(\gamma-(e_1-e_3)+b(T))'\sigma_t\epsilon_{t+1}^Q}(c(T) + d(T)'\left(\mu + AX_t + \sigma_t\sigma_t'\gamma\right) + d(T)'\sigma_t\epsilon_{t+1}^Q)\right] \\ &\quad = e^{a(T)+(-e_1+e_3+b(T))'(\sigma_t\sigma_t'\gamma+\mu+AX_t)} e^{\frac{1}{2}(-e_1+e_3+b(T))'\sigma\sigma_t'(-e_1+e_3+b(T))} \\ &\qquad \times (c(T) + d(T)'(\mu + AX_t + \sigma_t\sigma_t'\gamma) + d(T)'\sigma_t\sigma_t'(-e_1 + e_3 + b(T))).\end{aligned} \tag{E.4}$$

The last equality is obtained by employing Lemma A.1. Equating coefficients gives us the result. To obtain the initial conditions we directly evaluate $\mathrm{E}_t^Q(z_{t+1})$ using Lemma A.1.

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Notes

1. Penman (1991) argues that the RIM has implications for ratio analysis, particularly for the price-to-book and price-to-earnings ratios.
2. Zhang (1998) demonstrates that for a given level of earnings the relationship between book and equity value is ambiguous.
3. See Fama and French (2000) and Ou and Penman (1989), among others.
4. Nissim and Penman (2000) demonstrate that interest rates have predictive power to forecast future accounting returns.
5. Feltham and Ohlson (1999) use the notation ξ_{t+1} for the pricing kernel. We use the notation π_{t+1} as we save ξ_{t+1} to describe a component of π_{t+1} more in line with standard asset pricing notation. In certain situations the pricing kernel has a known form, such as the case of consumption-based asset pricing (Lucas, 1978), or with complete markets (Black and Scholes, 1973). Both of these functional forms are not valid here because we do not specify a representative agent who has utility over consumption and because markets are incomplete with respect to accounting information (market prices are not available for earnings or book values).
6. An equivalent representation of the pricing kernel is:

   $$m_t P_t = \mathrm{E}_t(m_{t+1} Z_{t+1})$$

   where m_{t+1} satisfies the equation:

   $$\pi_{t+1} = \frac{m_{t+1}}{m_t}.$$

   See Harrison and Kreps (1979).
7. Although this parameterization may look restrictive, it is very flexible. The strict positivity requirement is almost equivalent to specifying $\xi_{t+1} = \exp(f(\epsilon_{t+1}))$ for some function $f(\cdot)$. The sources of risk arise from the shocks of the driving variables in the economy, which are ϵ_{t+1}. For small variability in the driving factors, we can approximate the variable part of $f(\epsilon_{t+1})$ by a first-order approximation $\gamma' \sigma_t \epsilon_{t+1}$. This leads to the form of the pricing kernel in equation (5). This form of pricing kernel has been used in many financial applications including Duffie and Liu (2001), Bakshi and Chen (2001), Bekaert and Grenadier (2001), and others.
8. Strictly speaking Feltham and Ohlson (1999) consider a countable state space and a finite horizon. This can be generalized to most uncountable state spaces with some technical assumptions (see Harrison and Kreps, 1979), and applied to an infinite horizon sum by assuming transversality.
9. The process ξ_{t+1} converts the risk-neutral measure to the real measure. By definition the following relationship holds between the real measure and Q: $\mathrm{E}_t^Q[Z_{t+1}] = \mathrm{E}_t[\xi_{t+1} Z_{t+1}]$ for any $t+1$ measurable random variable Z_{t+1}. In technical terms, the process ξ_{t+1} is a Radon-Nikodym derivative of the real measure with respect to the risk-neutral measure Q. See Harrison and Kreps (1979).
10. We omit this proof as the derivation is very similar to the derivation of Proposition 3.1.
11. Note that g_t does not appear in the first term of the infinite sum. The reason is that at time $t+1$, the opportunity costs of the firm are $(e^{r_t} - 1)y_t$, which are known at time t. At time $t+2$ the opportunity costs of the firm are based on $y_{t+1} = y_t e^{g_t}$. This lag in the timing of g_t disappears if the model is formulated in continuous time. See Ang and Liu (1998).
12. The sample mean of X_t is $\bar{X}_t = (0.0817/4,\ 0.1366,\ 0.2773)'$ where the interest rate is annualized, but used as quarterly.

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