Market Segmentation and Software Security: Pricing Patching Rights

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June 2018

Abstract

The patching approach to security in the software industry has been less effective than desired. One critical issue with the status quo is that the endowment of "patching rights" (the ability for a user to choose whether security updates are applied) lacks the incentive structure to induce better security-related decisions. However, producers can differentiate their products based on the provision of patching rights. By characterizing the price for these rights, the optimal discount provided to those who relinquish rights and have their systems automatically updated in a timely manner, and the consumption and protection strategies taken by users in equilibrium as they strategically interact due to the security externality associated with product vulnerabilities, it is shown that the optimal pricing of these rights can segment the market in a manner that leads to both greater security and greater profitability. This policy greatly reduces unpatched populations and has a relative hike in profitability that is increasing in the extent to which patches are bundled together. Social welfare may decrease when automated patching costs are small because strategic pricing contracts usage in the market and also incentivizes loss-inefficient choices. However, welfare benefits when the policy either (i) greatly expands automatic updating in cases where it is minimally observed, or (ii) significantly reduces the patching process burden of those who most value the software.

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1   Introduction

Patchable – but unpatched – vulnerabilities consistently allow attackers to compromise widely used software such as systems provided by Microsoft, Adobe, and OpenSSL (US-CERT 2015). In fact, 64% of the top exploit samples in 2014 targeted patchable vulnerabilities from 2012 and prior (HP Security Research 2015). As much as 85% of targeted security attacks could be prevented by patching application and operating system vulnerabilities, in addition to employing whitelisting and access control strategies, according to the Canadian Cyber Incident Response Centre (CCIRC 2014).

However, many users do not promptly install patches. Less than 29% of Windows operating systems stay up to date according to OPSWAT, which collects data from software users through its security platform (OPSWAT 2014). As a typical example, Microsoft released a patch on March 14, 2017, following revelation of a critical vulnerability's existence (Microsoft 2017b). Two months later, despite the patch having been made available, the WannaCry ransomware attack struck over 200,000 computers across over 150 countries that had not yet patched (Lohr and Alderman 2017, Greenberg 2017). One month later and despite global media attention received by WannaCry, many users and organizations had still not patched, and, as a result, the NotPetya ransomware was able to spread by exploiting the same vulnerability (Microsoft 2017a).

In fact, many systems continue to remain unpatched long after patches are released. HP indicates in its recent *Cyber Risk Report 2015*, that "... the majority of exploits discovered by our teams attempt to exploit older vulnerabilities. By far the most common exploit is CVE-2010-2568, which roughly accounts for a third of all discovered exploit samples" (HP Security Research 2015). CVE-2010-2568, a Windows shell vulnerability that allows for remote code execution, was discovered in June of 2010. The patch for this vulnerability was released weeks later in August of 2010. Despite the patch being available for over six years, this vulnerability was *still* being exploited by attackers at the time of the report.

Massive financial losses are being incurred directly by those whose unpatched systems get compromised. The WannaCry attack cost billions of dollars due to productivity losses, mitigation efforts, paid ransom, and lost files (Berr 2017). In the case of NotPetya, a single organization, Maersk, incurred direct losses in the range of $200-$300 million because the attack forced it to halt shipping operations at 76 port terminals (Thomson 2017). Moreover, the presence of compromised systems imposes indirect losses on all users. These systems can be leveraged in other criminal

activities such as spam (Levchenko et al. 2011), distributed denial-of-service attacks (Fitzgerald 2015), and even the conducting of click fraud campaigns (Ingram 2015).

Observing today's cybersecurity attack landscape, the current patching process for security has been less effective than desired (August et al. 2014). Why aren't users patching? The growing reality is that security is not a technical problem, it's an economic one. Even though security patches are available, many users are not deploying them because it is not in their economic best interest to do so. For organizations, enterprise deployment of patches is a costly process. Extensive testing of patches in development and staging environments, roll-out of updates onto production servers, and final testing is both time consuming and resource intensive. Moreover, in aggregate there is a deluge of patches that system administrators must continuously monitor and process. For end users, the situation is regrettably similar because security patching is often not considered to be a priority. The deployment of updates and system rebooting is instead viewed as an inconvenience, particularly when users feel their own productivity is of greater concern.

Taking these costs into consideration, a software vendor can substantially increase security in an incentive-compatible way by encouraging improved user behavior. In particular, it can differentiate its software product by pricing *patching rights*.[1] Specifically, the vendor should charge users for the right to choose for themselves whether patches are installed or not installed on their systems. The status quo is that all users are endowed with patching rights, and a substantial portion of them elect not to patch as a result. By charging for patching rights, users who would otherwise have elected not to patch under the status quo must now examine whether it is worth paying for this right to remain unpatched. This decision is non-trivial as the expected security losses one would incur when retaining rights and remaining unpatched depends on the security behaviors of all other users in aggregate. On the flip side, by foregoing patching rights, users will have their systems automatically (and immediately[2]) patched by the vendor, and pay a different price. However, automatic deployment of patches also comes with risk because patches are not always stable. For example, some users encounter the "Blue Screen of Death" when applying a patch from Microsoft (Westervelt 2014, Sarkar and LeBlanc 2018). As another example of this risk, the cryptocurrency

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[1]Our work is based on the original idea that patching rights should be managed. Having been first introduced qualitatively in a perspectives piece (August et al. 2014), our paper is the first to formally model and analyze the value of patching rights.

[2]In our study, automatic updates are assumed to be applied either immediately or within a reasonably short time frame so as to reduce security risk. This type of automatic update is consistent with those provided by many leading software vendors (e.g., Microsoft, Oracle, Adobe, Google, etc.). There are some examples where firms manage automatic updates but do so in an delayed manner, such as was the case with Apple taking three months to patch the vulnerability exploited by the Flashback trojan (Hick 2012). This behavior is not in the spirit of the automated patching being modeled here.

exchange QuadrigaCX recently lost $14 million when inadequate patch testing failed to reveal that their process of transferring ether (a cryptocurrency) was incompatible with an Ethereum client software update (Higgins 2017). Considering these trade-offs, vendors have an incentive to target certain users with lower prices in equilibrium in exchange for their patching rights, hence these users will ultimately cause less of a security externality while benefitting from discounted software prices.

Many researchers are working toward improving security and understanding why patch availability alone has not led to very secure outcomes; our work aims to contribute toward progress on this front. In our model of security, users can choose whether to purchase a software product and additionally whether to remain patched, unpatched, or have their systems automatically updated in a timely manner by a software vendor. In this setting, examining the impact of optimally priced patching rights (PPR) on security, profitability, and overall economic value gives rise to insights that have promising practical implications to the software industry. Historically software companies have not differentiated on patching rights, but our insights may impel vendors to consider how a more profitable, more secure ecosystem can be achieved as a result. A discussion of how the essence of a PPR policy can be implemented as a component of a software vendor's overall versioning strategy follows our main results.

2   Literature Review

This work is related to three broad areas in the literature: (i) product differentiation and market segmentation, (ii) economics of information security, and (iii) economics of product bundling. With regard to the first stream, to the best of our knowledge, our paper is the first to examine how beneficial segmentation in software markets can be constructed by differentiation on software patching rights. More specifically, the focus is on a monopolist's product mix/line decisions when the quality-differentiated dimension concerns patching rights, as opposed to competition-driven product differentiation. Within the second area, this work is most closely related to a strand that studies the management of security patches. For the third area, our paper adds to a strand that examines the impact of mixed bundling on a monopolist's profit and social welfare.

Beginning with the first area, there is a well-developed literature in economics, marketing, operations management and information systems that examines the monopolist's problem of whether to offer and how to price quality-differentiated goods. In this vein, classic papers focus on char-

acterizing the general optimal non-linear price schedule that incentivizes each consumer to select the quality that is designed for her specific type (Mussa and Rosen 1978, Maskin and Riley 1984). Subsequent literature explored how these schedules react to various changes in consumer characteristics such as income dispersion and taste preferences (Moorthy 1984, Gabszewicz et al. 1986, Desai 2001, Villas-Boas 2009), firm characteristics such as production technology and marketing costs (Villas-Boas 2004, Debo et al. 2005, Netessine and Taylor 2007), and non-intrinsic characteristics such as strategic delays (Moorthy and Png 1992, Chen 2001, August et al. 2015). Within this body of work, our paper is most closely related to those that focus on consumer characteristics in that an automated patching version essentially modifies the costs incurred by consumers.

There are two critical contributions that our model adds to this portion of literature. First, the qualities that comprise product mix in our setting are endogenously determined in equilibrium by consumption decisions; this is a significant context-specific trait where users who do not patch their systems weakly reduce the quality of the software product for other users. Second, the price schedule is not based on the product that gets consumed but rather the *rights* retained by the consumer. Thus, our model admits interesting possibilities where consumers opt for the same rights (giving access to the same quality level) but then separate based on their own subsequent patching decisions (resulting in different effective qualities) – effective quality of the product is inclusive of security attacks which are endogenously determined by the strategic protection behaviors employed in equilibrium.

With regard to the second area of literature, our paper is close to work that studies the management of security patches. Researchers examine the timing of security patch release and its application (Beattie et al. 2002, Cavusoglu et al. 2008, Dey et al. 2015), vulnerability disclosure policy (Cavusoglu et al. 2007, Arora et al. 2008, Ransbotham et al. 2012), vendor patch policy (Lahiri 2012 and Kannan et al. 2013), and users' patching incentives (August and Tunca 2006, Choi et al. 2010). Our work is closest to the latter group of papers which construct models that endogenize users' patching decisions. Consistent with this work and models of vaccination (see, e.g., Brito et al. 1991), negative externalities stemming from unpatched behavior are modeled with further generalization to include risk that is independent of patching populations. While interesting in their own right, our model does not include other attack-related effects such as hiding effects (Gupta and Zhdanov 2012), zero-day vulnerabilities (August and Tunca 2011), or strategic attackers (Png and Wang 2009, Kannan et al. 2013). However, ours is the first to include both standard and automated patching options for users while also modeling the security externality. Inclusion of

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automated patching permits a characterization of the natural consumer market segmentation that arises in equilibrium as users strategically respond to security risk and expanded patching options. An understanding of equilibrium consumption and security behavior serves to inform how security enhancements should be marketed. Also, an automated patching option is the logical choice for the baseline product in a policy where patching rights are contracted, which is the focus of our work.

Toward addressing users' incentives, August and Tunca (2006) examine the efficacy of patching rebates. Patching rebates work by having the vendor subsidize patching costs in order to get more users to patch rather than remaining unpatched and contributing to security risk. August and Tunca (2006) show that these rebates can be very effective at improving behavior and security. They also show that if standard patching costs are large, it is not efficient to incentivize lower valuation users to incur these costs. One nice feature of the PPR policy is it does not require lower valuation users to incur these patching costs. Rather, it incentivizes them to have patches automatically deployed and instead incur potential system instability losses due to automated deployment. Since these losses tend to be lower, PPR works by incentivizing these users to engage in more appropriate and economical patching behaviors.

Finally, our work is related to a third body of literature examining a monopolist's decision regarding pure and mixed bundling. Under a PPR policy, software is unbundled from the right to decide whether or not to apply security patches.[3] There exists a well-developed literature on the bundling of physical products and information goods spanning the fields of economics, marketing, and information systems. Under moderate costs associated with automated patching, our proposed partial mixed bundling scheme (PPR) can simultaneously improve the software vendor's profit as well as security relative to the pure bundling alternative status quo. Several related works share the similar qualitative conclusion in which mixed bundling is favored over pure bundling and unbundled sales (see, e.g., Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989, Venkatesh and Kamakura 2003). These works show that bundling effectively extracts consumer surplus under various distributions of reservation values. In our work, partial mixed bundling when involving patching rights can possibly result in a slight decrease in social welfare, but it can also drive increases in social welfare depending on the quality of automated patching solutions and the extent to which security risk is reduced.

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[3]The status quo of providing software bundled with patching rights is in a sense pure bundling. PPR is partial mixed bundling: a bundle of software with patching rights and software alone without patching rights (Stremersch and Tellis 2002). Because patching rights have no standalone value, mixed bundling does not include the sale of patching rights.

In comparison to physical goods, information goods are typically assumed to have zero marginal costs, which enable the monopolist to bundle many information goods economically; this makes sense particularly when their valuations are correlated (Bakos and Brynjolfsson 1999). If two information goods provide highly asymmetric values to consumers, partial mixed bundling is optimal; the higher valuation good should not be sold separately not to cannibalize the sale of the bundle (Eckalbar 2010, Bhargava 2013). Idiosyncratic to our context, patching rights cannot be sold separately because they only have value to those who buy software. Unlike prior work on bundling, our model involves a security externality from unpatched software usage. Partial mixed bundling of two information goods has been shown to be optimal when only one good has a direct externality on consumer utility (Prasad et al. 2010). However, in our model, as more consumers purchase the automated patching version, other consumers become more willing to pay for the bundled version with patching rights due to the increased level of security.

3   Model Description and Consumer Market Equilibrium

3.1   Model

There is a continuum of consumers whose valuations of a software product lie uniformly on $\mathcal{V} = [0, 1]$. Consumers are exposed to security risks associated with the software's use. In particular, a vulnerability can arise in the software, in which case the vendor makes a security patch available to all users of the software. Because the security vulnerability can be used by malicious hackers to exploit systems, users who do not apply the security patch are at risk.

The vendor offers two options for users to protect their respective systems. In doing so, the vendor prices the software based on whether patching rights are granted to the consumer. Specifically, if a consumer elects to purchase the software and retain full patching rights, she pays the price $p \geq 0$. Having this right means that she can choose whether to patch the software or not patch the product and do so according to her own preferences. If she decides to patch the software, she will incur an expected cost of patching denoted $c_p > 0$. This *standard patching* cost accounts for the money and effort that a consumer must exert in order to verify, test, and roll-out patched versions of existing systems.[4]

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[4]Standard patching processes require considerable care, essentially coming down to labor costs associated with system administrators and developers spending time to complete all of the tasks in the patching process (Beres and Griffin 2009). Studies find that standard patching costs tend to be on the order of one thousand dollars per server (Bloor 2003, Forbath et al. 2005, Beres and Griffin 2009). Modeling the cost of standard patching as a constant is common in the literature that examines topics related to patching costs as can be seen in Beattie et al. (2002),

If she decides not to patch the software, then she faces the risk of an attack. Two classes of security losses are incurred: (i) those that are dependent on the size of the unpatched population of users, and (ii) those that are independent of the unpatched population size. For the dependent case, if she decides not to patch the software, the probability she is hit by a security attack is given by $\pi_s u$, where $\pi_s \in (0, 1]$ is the probability an attack appears and u is the size of the unpatched population of users.[5] This reflects the negative security externality imposed by unpatched users of the software. If she is successfully attacked, then she will incur expected security losses that are positively correlated with her valuation. For simplicity, the correlation is assumed to be of first order, i.e., the loss that a consumer with valuation v suffers if she is hit by an attack is $\alpha_s v$ where $\alpha_s > 0$ is a constant. The quantity $\pi_s \alpha_s$ is referred to as *dependent risk* throughout the paper. The dependent case directly captures any attack that spreads through vulnerable populations and is agnostic to the specific attack vector or mechanism by which spreading occurs.

The dependent case also indirectly captures any type of security attack where the incentives of the malicious individual for constructing the attack is positively related to the unpatched population size. For example, if large vulnerable populations are more attractive to hackers because it becomes easier to penetrate hosts or the return on their efforts becomes higher when infecting more hosts, then the dependent case applies. On the other hand, unpatched users can also face risk from attacks that are independent of the size of the unpatched population. This class can include targeted attacks and other forms of background risk. Using analogous notation, the likelihood of an independent attack is denoted by $\pi_i \in (0, 1]$, and similarly $\pi_i \alpha_i$ is referred to as *independent risk*, where $\alpha_i > 0$ is a constant.

If the consumer instead elects to purchase the software and relinquish patching rights, she pays the price δp, where $\delta \geq 0$. In this case, the vendor retains full control over patching the software and will automatically and immediately do so to better protect the user population.[6] From an implementation point of view, this software version would not give users much or any control over patch deployment (e.g., the typical options can be grayed out in this version). The user incurs a cost of *automated patching*, $c_a > 0$, which is associated with both inconvenience and configuration

August and Tunca (2006), Choi et al. (2010) and Cavusoglu et al. (2008).

[5]The size of the unpatched population u is determined by the consumer strategies in equilibrium. Therefore, by the definition of $\mathcal{V}$, $u \in [0, 1]$.

[6]This is fairly easy to enforce in interconnected networks. For example, vendors such as Adobe and Matlab enforce real-time license checks for their subscription-based offerings. While it is always possible to circumvent protections, most paying customers are unlikely to break the license agreement. Our model assumes immediate patch deployment for simplicity. The essence of our results only require that patchers, whether automated or standard, complete tasks in a relatively timely manner that distinguishes them from those who do not patch.

of the system to handle automatic deployment of security patches.[7] There is always some risk associated with an automatically deployed patch causing a user's system to become unstable or even crash because a vendor cannot test for compatibility of the patch with every possible user system configuration. This probability that the automated patch is problematic is denoted by $\pi_a \in (0, 1]$. The loss associated with an automated patch deployment failure is again positively correlated with her valuation. Assuming first-order correlation, denoted $\alpha_a > 0$, her expected loss associated with automated patching is given by $\pi_a \alpha_a v$.[8]

Each consumer decides whether to buy, B, or not buy, NB. Similarly, for her patching decision, she chooses one of patch, P, not patch, NP, and automatically patch, AP. In order to choose P or NP, she must pay the price p to retain patching rights. By choosing AP, she delegates patching rights to the vendor and pays the price δp. The consumer action space is then given by $S = (\{B\} \times \{P, NP, AP\}) \cup (NB, NP)$. In a consumer market equilibrium, each consumer maximizes her expected utility given the equilibrium strategies for all consumers. For a given strategy profile $\sigma : \mathcal{V} \to S$, the expected utility for consumer v is given by:

$$U(v, \sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (B, P); \\ v - p - \pi_s \alpha_s u(\sigma) v - \pi_i \alpha_i v & if \quad \sigma(v) = (B, NP); \\ v - \delta p - c_a - \pi_a \alpha_a v & if \quad \sigma(v) = (B, AP); \\ 0 & if \quad \sigma(v) = (NB, NP), \end{cases} \tag{1}$$

where

$$u(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v) = (B, NP)\}} \, dv. \tag{2}$$

To avoid trivialities and without loss of generality, the parameter space is reduced to $c_p, c_a \in (0, 1)$ and $\pi_a \alpha_a \in (0, 1 - c_a)$, which ensures automated patching is economical.

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[7]Our model can examine any relationship between c_p and c_a. For example, it can capture the commonly observed situation in which users are choosing between: (i) completing all tasks associated with the rigorous, standard patching approach and incurring c_p, or (ii) doing the bare minimum to deploy patches automatically without verification and incurring a lower cost, $c_a < c_p$, related to deployment. The model can also handle situations where $c_a \geq c_p$, to study scenarios in which users aim to achieve all tasks associated with standard patching but in an automated manner.

[8]The loss factor $\pi_a \alpha_a$ captures in expectation major patch failures that would lead to severe backlash against the vendor. An increased likelihood of such events is represented by a higher $\pi_a \alpha_a$, which will affect the value of a PPR policy.

3.2 Consumer Market Equilibrium

Before examining how patching rights should be priced, it is necessary to characterize how consumers segment across strategies for an arbitrary set of prices in equilibrium. Endogenous determination of the security externality that results from usage and patching decisions complicates the situation. The consumer with valuation v selects an action that solves the following maximization problem:

$$\max_{s \in S} \quad U(v, \sigma), \tag{3}$$

where the strategy profile σ is composed of σ_{-v} (which is taken as fixed) and the choice being made, i.e., $\sigma(v) = s$. Her optimal action that solves (3) is denoted $s^*(v)$. The equilibrium strategy profile is denoted σ^*, and it satisfies the requirement that $\sigma^*(v) = s^*(v)$ for all $v \in \mathcal{V}$.

Lemma 1 *There exists a unique equilibrium consumer strategy profile σ^* that is characterized by thresholds v_b, v_a, $v_p \in [0,1]$. For each $v \in \mathcal{V}$, it satisfies either*

$$\sigma^*(v) = \begin{cases} (B, P) & if \quad v_p < v \le 1\,; \\ (B, NP) & if \quad v_b < v \le v_p\,; \\ (B, AP) & if \quad v_a < v \le v_b\,; \\ (NB, NP) & if \quad 0 \le v \le v_a\,, \end{cases} \tag{4}$$

or

$$\sigma^*(v) = \begin{cases} (B, P) & if \quad v_p < v \le 1\,; \\ (B, AP) & if \quad v_a < v \le v_p\,; \\ (B, NP) & if \quad v_b < v \le v_a\,; \\ (NB, NP) & if \quad 0 \le v \le v_b\,. \end{cases} \tag{5}$$

Lemma 1 establishes that if a population of patched consumers arises in equilibrium, it will consist of a segment of consumers with the highest valuations. These consumers prefer to shield themselves from any valuation-dependent losses born when either remaining unpatched (security losses) or using automated patching (instability losses). Importantly, this segment need not arise (e.g., the upper threshold satisfies $v_p = 1$ in cases where valuation-dependent losses are smaller than patching costs). The middle segments, composed of consumers who elect for automated patching or to remain unpatched, can be ordered either way depending on the relative strength of the losses under each strategy.

4 Pricing Patching Rights

Since the value of patching is most applicable under higher security risk, our study centers on regions such that patching is worthwhile (i.e., either independent risk or dependent risk is reasonably high). There are several merits to begin analysis with high independent risk. First, based on user incentives, this case is inherently simpler and ultimately admits closed-form solutions of the consumer market thresholds and prices that arise in equilibrium. This helps build intuition into how a PPR policy tends to affect equilibrium behaviors. Second, the impact of high independent risk is similar in nature to that of high dependent risk in that both tend to reduce unpatched populations as users become unwilling to bear higher risk. In this light, certain limit effects on thresholds and profitability will be the same and can be characterized more easily in a simplified setting. Third, examination of this case underscores why capturing dependent risk is essential to a comprehensive understanding of interdependent security settings, which propels the remainder of the paper.

4.1 High Independent Risk

For convenience, parameter sets satisfying $\pi_i \alpha_i > 1$ are examined as a proxy for the case of high independent risk. Section A of the Appendix provides a complete characterization of the parameter conditions and thresholds for each possible consumer market structure that can arise in equilibrium. There are three possible structures, with the two most relevant to the current discussion having threshold orderings given by $0 < v_a < v_p < 1$ and $0 < v_p < 1$. These two structures obtain under broad conditions which, in turn, can be satisfied under equilibrium pricing decisions. Under high independent risk, no user will elect to be unpatched, which is to say there is no security externality in equilibrium.

Turning toward equilibrium pricing, the vendor's profit function is given by

$$\Pi(p, \delta) = p \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v|p,\delta) \in \{(B,NP),(B,P)\}\}} dv + \delta p \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v|p,\delta)=(B,AP)\}} dv, \tag{6}$$

noting that marginal costs are assumed to be negligible for information goods. Given an interest in determining the benefit of optimally pricing the *right* for a user to determine whether or not to install patches on her system, it is useful to first present a characterization of the equilibrium when this right is not priced. In this reference case, referred to throughout the paper as the *status quo*, standard practice for the industry is that $\delta = 1$ such that the price is the same regardless

of patching behavior. In this case, the vendor chooses a price p for the software by solving the following problem:

$$\max_{p \in [0, \infty)} \quad \Pi(p, \delta)$$

$$\text{s.t.} \qquad (v_b, v_a, v_p) \text{ are given by } \sigma^*(\cdot \mid p, \delta), \tag{7}$$

$$\delta = 1.$$

Given a price p^* that solves (7), the profit associated with this optimal price are denoted by $\Pi_{SQ} \triangleq \Pi(p^*, 1)$.

Lemma 2 (Status Quo) *Suppose that $\pi_i \alpha_i > 1$ and $\delta = 1$ (i.e., when patching rights are not priced).*

(i) *If $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then*

$$p^* = \frac{1 - \pi_a \alpha_a - c_a}{2}, \tag{8}$$

and σ^* is characterized by $0 < v_a < v_p < 1$ such that the lower tier of users prefers automated patching.

(ii) *On the other hand, if $c_a \geq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then*

$$p^* = \frac{1 - c_p}{2}, \tag{9}$$

and σ^* is characterized by $0 < v_p < 1$ such that there is no user of automated patching in equilibrium.

Lemma 2 presents the equilibrium behavior under the status quo reference case. Part (i) establishes that as long as the cost of standard patching (c_p) and automated patching (c_a) satisfy conditions where c_a is moderate, both standard patching and automated patching are observed strategies in equilibrium. On the other hand, as c_a increases to a higher level, only standard patching is observed. Lemma 2 highlights that high independent risk essentially squeezes out unpatched behaviors, leading to the absence of a security externality in equilibrium. It is useful to understand the role of PPR in this context as it will serve as a contrastable reference point for when externalities become a driving force.

When patching rights are priced, the vendor jointly selects a price and multiplier (p, δ) to

maximize his profits. His pricing problem is formulated as follows:

$$\max_{(p,\delta)\in[0,\infty)^2} \quad \Pi(p,\delta) \tag{10}$$

$$\text{s.t.} \qquad (v_b, v_a, v_p) \text{ are given by } \sigma^*(\cdot|p,\delta).$$

Under the optimal choices (p^*, δ^*) which solve (10), the optimal profit under priced patching rights is denoted by $\Pi_P \triangleq \Pi(p^*, \delta^*)$.[9]

Lemma 3 (PPR) *Suppose that $\pi_i\alpha_i > 1$ and patching rights are priced by the vendor.*

(i) *If $c_p - \pi_a\alpha_a < c_a < c_p(1 - \pi_a\alpha_a)$, then*

$$p^* = \frac{1 - c_p}{2}, \tag{11}$$

$$\delta^* = \frac{1 - c_a - \pi_a\alpha_a}{1 - c_p}, \tag{12}$$

and σ^* is characterized by $0 < v_a < v_p < 1$ such that the lower tier of users prefers automated patching.

(ii) *On the other hand, if $c_a \geq c_p(1 - \pi_a\alpha_a)$, then*

$$p^* = \frac{1 - c_p}{2}, \tag{13}$$

$$\delta^* = 1, \tag{14}$$

and σ^* is characterized by $0 < v_p < 1$ such that there is no user of automated patching in equilibrium.

Lemma 3 formally establishes that PPR can greatly expand the region of the parameter space on which automated patching is observed, relative to the status quo. Specifically, when the condition $1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a} \leq c_a < c_p(1 - \pi_a\alpha_a)$ is satisfied, then automated patching arises $(0 < v_a < v_p < 1)$ when patching rights are priced whereas it does not arise $(0 < v_p < 1)$ in the status quo. When part (i) of Lemma 3 is satisfied, then patching rights are priced in a way that consumers who select automated patching in equilibrium form the lower tier of the consumer market. In equilibrium, the vendor charges the "premium" $p^*(1 - \delta^*)$ for patching rights; alternatively, $p^*(1 - \delta^*)$

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[9]Going forward, subscripts "SQ" and "P" indicate a particular measure refers to the outcome under the status quo and under a PPR policy, respectively, for consistency.

can be considered the "discount" given to users who agree to have their systems automatically updated to reduce security risk.

By (11) and (12), it is easy to see that the premium charged for patching rights is only greater than the cost of automated patching (c_a) when standard patching costs (c_p) are small enough. To understand why, the software vendor has an incentive to charge a high price for his software when standard patching costs are small. One can think of his product as being better software that is easily maintained via a cost-effective, rigorous patching process. Therefore, the vendor can achieve a sizable user population, most of which elects for standard patching, even when charging a high price. Because of the attractiveness of standard patching, it is necessary to provide a significant discount to incentivize users to prefer the automated patching option. This is reflected in (12); the optimal price multiplier (δ^*) decreases as standard patching costs (c_p) decrease. From the other perspective, the patching rights premium $p^*(1 - \delta^*)$ is substantial and can be an incentive-compatible option only for high-valuation users. Low-valuation users necessarily find the patching rights premium too high to bear and instead opt for automated patching in equilibrium.

Proposition 1 *When $\pi_i \alpha_i > 1$, if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, the relative increase in profitability of pricing patching rights is given by*

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a \alpha_a)(c_a - c_p + \pi_a \alpha_a)^2}{\pi_a \alpha_a (1 - c_a - \pi_a \alpha_a)^2}, \tag{15}$$

where

$$\Pi_{SQ} = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}. \tag{16}$$

Proposition 1 formally establishes the extent to which a PPR policy can increase profitability for the vendor. In the context of our overall study, what is important to emphasize here is that the vendor can have strong incentives to leverage an automated patching version toward discriminatory purposes. In particular, under high independent risk, Lemmas 2 and 3 establish that no unpatched usage arises in equilibrium and so it is not the case that the PPR policy being employed aims to reduce the security externality. The segmentation behavior seen here solely targets extraction of surplus from high-valuation users by inducing them to pay the patching rights premium. These users have much to lose in the event of any system failure occurring due to patch instability, and thus are willing to pay to retain control and continue to exercise diligence in their patching processes.

Users whose valuations are not high will find it incentive-compatible to relinquish patching rights. In fact, a sizable segment of consumers switch from standard patching towards automated

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patching when patching rights are priced. As the proofs of part (i) of Lemmas 2 and 3 establish, the user type indifferent between using automated patching and not even buying the software (v_a) is identical both in the status quo and under PPR. Viewed in that light, the pricing of patching rights does not expand usage in the market, instead only serving to encourage some users to make less loss-efficient security choices yet benefitting vendor profitability.

This proposition also shows that a PPR policy always outperforms a mandated automated patching policy. The reason is because mandating automated patching is a special case of a priced patching rights policy. Specifically, the decision problem when mandating automatic patching can be formulated in the same manner as the PPR problem, subject to an additional constraint that the price of patching rights is prohibitively high (i.e., $p = 1$). As seen in the utility function given by (1), setting such a price makes the strategies of being unpatched (B, NP) and using standard patching processes (B, P) infeasible to consumers, and the vendor chooses a price multiplier (δ) to maximize profits with all consumers now only considering automated patching (B, AP). Because the price $p = 1$ is feasible but never chosen in the original problem, mandating automatic patching leads to strictly lower profits.

A vendor's patch release frequency impacts PPR's relative profitability. A frequent patch release policy imposes additional burden on those who follow a standard patching policy. In our model abstraction, higher frequency corresponds to higher standard patching costs (i.e., higher c_p). By equation (15), the relative profitability of PPR is decreasing in standard patching costs. Because higher standard patching costs naturally incentivize users to shift toward automated patching usage rather than unpatched usage (due to high independent risk $\pi_i \alpha_i$), the upside of PPR becomes limited. More frequently released patches can also reduce costs associated with patch instability because problems are much easier to diagnose when scope is narrower. In (15), relative profitability is similarly decreasing as instability risk ($\pi_a \alpha_a$) decreases. Overall, the relative value of PPR is generally higher when patches tend to be bundled together.

Proposition 1 highlights the discriminatory forces at work when the vendor can separately price an automated patching version of his product without being concerned about security externalities. On the other hand, equilibrium consumer market outcomes marked by no user being unpatched call attention to the source of the risk. In particular, one might ponder why independent risk ($\pi_i \alpha_i$) is high if nobody is unpatched. Even if only a few users out of a large population were unpatched, should we expect them to face high risk? This line of thought suggests that security risk and the size of the unpatched population may naturally have some dependence which is the focus of the

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next section.

4.2 Low Independent Risk

When independent risk is more moderate in nature (i.e., explicitly bounded above), equilibrium outcomes now differ under high dependent risk. In contrast to the prior section, in equilibrium there exists a segment of unpatched users regardless of dependent risk being high. The characterization of the thresholds that emerge in the equilibrium consumer market structure become significantly more complex, satisfying a nonlinear system of equations. Therefore, asymptotic analysis is employed, which is commonly found in microeconomic studies.[10]

As before, it is helpful to first characterize the consumer market equilibrium when patching rights are freely included.

Lemma 4 (Status Quo) *There exists $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $\pi_i \alpha_i < \min\left[\frac{c_p \pi_a \alpha_a}{1 + c_p - c_a}, \frac{c_p}{1 + c_p}\right]$, $\delta = 1$ (i.e., when patching rights are not priced),*

(i) if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then

$$p^* = \frac{1}{2}(1 - \pi_a \alpha_a - c_a) + \frac{2c_a^2(\pi_a \alpha_a - 1)((\pi_a \alpha_a - 1)(2\pi_a \alpha_a - \pi_i \alpha_i - 1) + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3))}{(-\pi_a \alpha_a + c_a + 1)^3 \pi_s \alpha_s} + K_a,$$
(17)

and σ^* is characterized by $0 < v_b < v_a < v_p < 1$ such that the lower tier of users remain unpatched and the middle tier prefers automated patching.

(ii) On the other hand, if $c_a > 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then

$$p^* = \frac{1 - c_p}{2} - \frac{2c_p^2(1 - 3c_p + \pi_i \alpha_i (1 + c_p))}{(1 + c_p)^3 \pi_s \alpha_s} + K_b,$$
(18)

and σ^* is characterized by $0 < v_b < v_p < 1$ such that there is no user of automated patching in equilibrium.[11]

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[10]Its use can be expected here due to the complexity of the game and corresponding equilibrium characterization (some examples of studies using asymptotic analysis include Li et al. 1987, Laffont and Tirole 1988, MacLeod and Malcomson 1993, Pesendorfer and Swinkels 2000, Muller 2000, Tunca and Zenios 2006, August and Tunca 2006, Pei et al. 2011 among many others). Miller (2006) and Cormen et al. (2009) provide comprehensive treatments of the mathematical foundation underlying asymptotic analysis. Due to model complexity in this region, some boundaries do not have explicit functional forms. However, the objective of the analysis is the identification of regions of applicability in terms of parameter characteristics, which is the focus of our formalized results.

[11]The existence of $\tilde{\alpha}_s$ is proven in the Appendix. The characterization of constants denoted by K and enumerated by a subscript are similarly provided.

Part (i) of Lemma 4 provides the reference case when the cost of automated patching is relatively moderate. An immediate observation is that when patching rights are free as in the status quo, the consumer segment whose equilibrium strategy is to use automated patching is always the middle tier. This occurs because when a user compares an automated patching strategy to an unpatched strategy, the price is the same for both options. Therefore, the former strategy is preferred to the latter as long as the condition $v[\pi_s \alpha_s u(\sigma^*) + \pi_i \alpha_i - \pi_a \alpha_a] > c_a$ is satisfied. Notably, this condition is monotone in v which is to say that if it is satisfied for any user with valuation v, it is also satisfied for any user with a valuation higher than v. As a result, the automated patching segment of users always form the middle tier. Contrasting this to the previous section, under high independent security risk the automated patching segment formed the lowest tier. This cannot happen in the current region when patching rights are endowed.[12]

These observations highlight an important potential impact of a PPR policy; if the premium charged for patching rights, $p(1 - \delta)$, is greater than the cost of automated patching (c_a) and the unpatched population, $u(\sigma^*)$, decreases enough in equilibrium, then the lower tier can instead be composed of users who strategically choose automated patching. In this sense, a PPR policy can fundamentally change segmentation behavior in the consumer market, which in turn can have a significant impact on security and profitability. The following lemma formalizes the equilibrium strategies under PPR.

Lemma 5 (PPR) *Suppose that $\alpha_s > \tilde{\alpha}_s$, $\pi_i \alpha_i < \min \left[\frac{c_p \pi_a \alpha_a}{1 + c_p - c_a}, \frac{c_p}{1 + c_p} \right]$, and that patching rights are priced by the vendor.*

(i) If $c_a < \min \left[\pi_a \alpha_a - c_p, c_p(1 - \pi_a \alpha_a) \right]$, then

$$p^* = \tilde{p} + \left(2\pi_a \alpha_a c_p \left(\pi_i \alpha_i c_a^2 + c_a(-\pi_a \alpha_a \pi_i \alpha_i + 3\pi_a \alpha_a c_p - 2c_p \pi_i \alpha_i) + c_p(\pi_a \alpha_a(\pi_a \alpha_a + \pi_i \alpha_i) + \right. \right.$$

$$\left. \left. c_p(\pi_i \alpha_i - 3\pi_a \alpha_a)) \right) \right) \left(\pi_s \alpha_s(-\pi_a \alpha_a + c_a - c_p)^3 \right)^{-1} + K_c, \quad (19)$$

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[12]Under low independent risk whenever a standard patching population arises in equilibrium, there must also be a population of unpatched users. Otherwise, a standard patching user would deviate to being unpatched and bear no risk.

$$\delta^* = \tilde{\delta} - \Bigg(4c_p\pi_a\alpha_a(\pi_a\alpha_a + c_a - 1)\big(\pi_i\alpha_i c_a^2 + c_a(-\pi_a\alpha_a\pi_i\alpha_i + 3\pi_a\alpha_a c_p - 2\pi_i\alpha_i c_p) +$$

$$c_p(\pi_a\alpha_a(\pi_a\alpha_a + \pi_i\alpha_i) + c_p(\pi_i\alpha_i - 3\pi_a\alpha_a)))\big) \Bigg) \Bigg(\pi_s\alpha_s(c_p - 1)^2(\pi_a\alpha_a - c_a + c_p)^3 \Bigg)^{-1} + K_d,$$

$$(20)$$

and σ^* is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing, where $\tilde{p} = \frac{1-c_p}{2}$ and $\tilde{\delta} = \frac{1-\pi_a\alpha_a - c_a}{1-c_p}$, such that the lower tier of users prefer automated patching and the middle tier remains unpatched;

(ii) if $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, then

$$p^* = \tilde{p} + \frac{c_a(c_a - c_p\pi_a\alpha_a + c_p)(c_a(1 - \pi_i\alpha_i) + (1 - \pi_a\alpha_a)(-\pi_i\alpha_i + 2c_p - 1))}{\pi_s\alpha_s(1 + c_a - \pi_a\alpha_a)^3} + K_e, \quad (21)$$

$$\delta^* = \tilde{\delta} - \Bigg(2c_a\big(c_a^2 + (\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_p^2 - 2c_p\pi_a\alpha_a)\big)(c_a(\pi_i\alpha_i - 1) + (\pi_a\alpha_a - 1)(-\pi_i\alpha_i + 2c_p - 1)) \Bigg)$$

$$\Bigg((c_p - 1)^2\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^3 \Bigg)^{-1} + K_f, \quad (22)$$

and σ^* is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing such that the lower tier of users remains unpatched and the middle tier prefers automated patching;

(iii) if $c_a > c_p(1 - \pi_a\alpha_a)$, then

$$p^* = \tilde{p} - \frac{2c_p^2(1 - 3c_p + \pi_i\alpha_i(1 + c_p))}{(c_p + 1)^3\pi_s\alpha_s} + K_g, \quad (23)$$
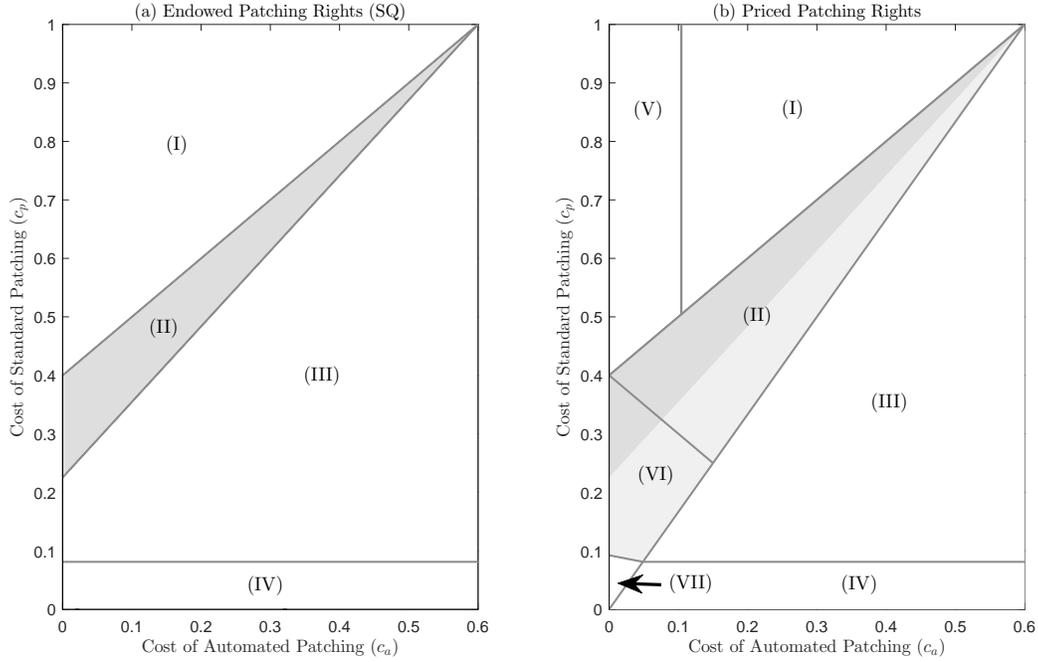
$$\delta^* = 1, \quad (24)$$

and σ^* is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

Lemma 5 demonstrates that a restructuring of the consumer market can indeed be the equilibrium outcome when patching rights are priced. Specifically, if the patching costs are small such that part (i) of Lemma 5 is satisfied, then the equilibrium patching rights are priced in a way that consumers who select automated patching in equilibrium form the lower tier of the consumer market. This outcome more closely resembles the structure that emerges in part (i) of Lemma 3, with a similar driving force. Specifically, small standard patching costs prompt a high patching rights premium that low-valuation users are unwilling to assume.

On the other hand, when the patching rights premium is limited, the equilibrium price and discount induce a consumer market structure that remains consistent with what unfolds under the status quo, but differs from the case of high independent risk. Part (ii) of Lemma 5 shows that this structure is characterized by the threshold ordering $0 < v_b < v_a < v_p < 1$, which matches the ordering in the first part of Lemma 4. Thus, under both the status quo and under PPR, the middle tier is incentivized to select the automated patching option in equilibrium. Finally, part (iii) of Lemma 5 establishes that if the loss factor associated with automated patching losses (i.e., patch instability) becomes too large, the vendor is best off not providing a discount in exchange for patching rights. Rather, he prices the software such that users do not elect for automated patching in equilibrium.

Figure 1 demonstrates how pricing patching rights significantly affects the consumer market structures that are obtained in equilibrium. Each panel illustrates the consumer market structure threshold characterization that obtains in equilibrium as a function of standard patching (c_p) and automated patching (c_a) costs. Panel (a) shows that four possible market structures can arise in the status quo under conditions in which both independent risk ($\pi_i \alpha_i = 0.15$) and patch instability risk ($\pi_a \alpha_a = 0.4$) are reasonably low such that unpatched and automated patching behaviors can be observed. When standard patching costs are relatively high as in Region (I), it becomes impractical to conduct standard patching processes. In this case, even high-valuation consumers are willing to bear the patch instability risk associated with automated patching, hence the consumer market structure is characterized by the absence of a standard patching segment (i.e., a threshold ordering of $0 < v_b < v_a < 1$). At the other extreme, in Regions (III) and (IV) where standard patching costs are relatively low, automated patching is not observed in equilibrium. The consumer market structure has a threshold ordering of either $0 < v_b < v_p < 1$ or $0 < v_p < 1$ due to the cost effectiveness of standard patching processes (the absence of v_a is akin to unobserved automated patching). Finally, when standard patching and automated patching costs have an intermediate relationship as seen in Region (II), the vendor's pricing leads to an equilibrium characterized by all user segments being represented. In particular, the threshold ordering that arises in equilibrium is $0 < v_b < v_a < v_p < 1$ where the automated patching segment notably consists of the middle tier of valuations.

When patching rights are priced, there are two distinct changes to user behavior that are illustrated in panel (b) of Figure 1. First, the region over which automated patching is preferred by some consumers in equilibrium significantly expands under PPR. Panel (b) uses grayscale to illustrate how Region (II) expands and splits into two sub-regions; dark gray delineates the common region across both panels, and light gray delineates the expansion under PPR. For this to occur,

Figure 1: Characterization of equilibrium consumer market structures under endowed (SQ) and priced patching rights (PPR) policies for high dependent risk. Panel (a) illustrates the endowed case or status quo, whereas panel (b) illustrates the PPR policy. Region labels describe the consumer segments that arise in each region in order of increasing consumer valuations (from left to right). Grayscale highlights the market structure with all segments represented, its expansion under PPR, and the reordering of segments that occurs, i.e., Region (VI). Independent security risk ($\pi_i\alpha_i = 0.15$) and automated patch instability risk ($\pi_a\alpha_a = 0.4$) are chosen to ensure unpatched and automated patching behaviors are present.

the region of the parameter space over which automated patching behavior is absent under the status quo shrinks upon pricing patching rights. This is easily visualized by Regions (III) and (IV) decreasing in size when moving from panel (a) to panel (b). The expansion of automated patching behavior is a critical effect of PPR because it goes hand-in-hand with a decreased unpatched population which helps to reduce security risk.

Second, a PPR policy can create entirely new market structures that are not observed under the status quo. Region (VI) of panel (b) illustrates a region of the parameter space in which the

threshold characterization is now given by the ordering $0 < v_a < v_b < v_p < 1$ where the automated patching segment consists of the lower tier of valuations. Specifically, when patching rights are endowed as in the status quo, it is not possible to get low-valuation users to adopt the automated patching solution; instead, they remain as unpatched, externality-contributing users. Under PPR and low standard patching costs (hence a high premium as discussed previously) and low automated patching costs, it becomes no longer incentive-compatible for these users to remain being unpatched. However, this behavior can change as c_a increases. This is illustrated as a shift from Region (VI) to either Region (II) or Region (III) in panel (b) of Figure 1 where lower valuation users once again prefer to be unpatched.

Building on this understanding of how PPR affects usage, the following proposition provides greater clarity into the strategic behavior underlying the vendor's pricing as well as its impact on security.

Proposition 2 *There exists a bound $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$ and $\pi_i\alpha_i < \min\left[\frac{c_p\pi_a\alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$, then a PPR policy can improve profits while reducing the security externality generated by unpatched users as compared to when patching rights are not priced. When $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)\right]$, the relative increase in profitability is given by*

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2} + \Bigg((\pi_a\alpha_a - c_a + c_p)^2 M - 4\pi_a\alpha_a(1 - \pi_a\alpha_a)(1 - \pi_a\alpha_a - c_a)$$

$$(-\pi_a\alpha_a + c_a + c_p)\left(\pi_a\alpha_a((c_a + 2)c_p + c_a) + c_a(1 - c_p)(c_p - c_a) - 2c_p(\pi_a\alpha_a)^2\right)(c_a - c_p(1 - \pi_a\alpha_a)) +$$

$$4\pi_i\alpha_i(\pi_a\alpha_a - 1)(-\pi_a\alpha_a + c_a + 1)(-\pi_a\alpha_a + c_a - c_p)\left((\pi_a\alpha_a)^3(\pi_a\alpha_a)^2(c_a + c_p + 1) - \pi_a\alpha_a(c_a - 1)(c_a + c_p) +\right.$$

$$c_a(c_a - c_p)^2\big)(c_a + c_p(\pi_a\alpha_a - 1))\Bigg)\left(\pi_a\alpha_a\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2(\pi_a\alpha_a + c_a - 1)^3(\pi_a\alpha_a - c_a + c_p)^2\right)^{-1} + K_h \, ,$$

$$(25)$$

where

$$M = 4c_a(\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_a - c_p)\left((\pi_a\alpha_a + c_a)\left(\pi_a\alpha_a(2 - \pi_a\alpha_a) + c_a(\pi_a\alpha_a - 2) + 2c_p(\pi_a\alpha_a - 1)^2\right) - \pi_a\alpha_a\right) \, ,$$

$$(26)$$

$$\Pi_{SQ} = \frac{(1 - c_a - \pi_a\alpha_a)^2}{4(1 - \pi_a\alpha_a)} + \frac{c_a(1 - c_a - \pi_a\alpha_a)((1 - \pi_a\alpha_a)(\pi_a\alpha_a - \pi_i\alpha_i) + c_a(2 - \pi_a\alpha_a - \pi_i\alpha_i))}{(1 + c_a - \pi_a\alpha_a)^2\pi_s\alpha_s} + K_i \, ,$$

$$(27)$$

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and the reduction in the size of the unpatched population is given by

$$u_{SQ}^* - u_P^* = \frac{\left(c_a^2(\pi_a\alpha_a - 2) + c_a(\pi_a\alpha_a + c_p(2 - 3\pi_a\alpha_a)) + \pi_a\alpha_a(\pi_a\alpha_a - 1)(c_p - \pi_a\alpha_a)\right)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)(\pi_a\alpha_a - c_a + c_p)} + K_j \, . \tag{28}$$

When $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, the relative increase in profitability is given by

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2} + \Bigg(M + (-\pi_a\alpha_a + c_a + 1)\big(4\pi_i\alpha_i c_a(\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_a - c_p)$$
$$(c_a + c_p(\pi_a\alpha_a - 1))\big)\Bigg)\Bigg(\pi_a\alpha_a\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2(\pi_a\alpha_a + c_a - 1)^3\Bigg)^{-1} K_k \, , \quad (29)$$

where Π_{SQ} is given by (27) and the reduction in the size of the unpatched population is given by

$$u_{SQ}^* - u_P^* = \frac{(1 - \pi_a\alpha_a)(\pi_a\alpha_a + c_a - c_p)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)} + K_l \, . \tag{30}$$

Proposition 2 highlights an important message from our study: software vendors should consider differentiation of their products based on patching rights. Simply providing patches for security vulnerabilities of software to users as a security strategy has not worked well. In many cases, it leads to large unpatched user populations as these users determine it is not in their best interest to patch. The externality they cause is detrimental to security and to the vendor's profitability. Proposition 2 formally establishes that the proper pricing of patching rights can increase profits for vendors to an extent characterized in (25) and (29), and simultaneously reduce the size of the unpatched population. Thus, there are potentially large economic and security benefits associated with a PPR policy, and this can be an important paradigm shift for the software industry.

Product differentiation is an important topic studied in economics and marketing, and the versioning of information goods has further nuanced findings (Bhargava and Choudhary 2001, 2008, Johnson and Myatt 2003). In particular, for these goods which have a negligible marginal cost of reproduction, a software vendor finds it optimal to release only one product (no versioning) when consumers heterogeneous taste for quality is uniformly distributed. In such a case, cannibalization losses outweigh differentiation benefits. In the current work, Proposition 2 demonstrates that if the versioning is instead on patching rights, a versioning strategy is once again optimal for the vendor. In this case, the software vendor can profitably benefit by increasing the price of the version with patching rights (p^*) relative to the price point under the status quo. By doing so,

while concurrently decreasing the price of the version without patching rights ($\delta^* p^*$), there are several effects as consumers strategically respond. First, a higher p^* puts pressure on any user who would be unpatched under the status quo to reconsider the trade-off. Under the status quo equilibrium consumer market structure (i.e., $0 < v_b < v_a < v_p < 1$), the unpatched users form the lower tier of the consumer market (those with valuations between $[v_b, v_a]$). Because patching rights are endowed to all users under the status quo, these users remain unpatched and contribute to a larger security externality. Under PPR, a higher p^* makes it now more expensive to remain in the population as an unpatched user causing this externality. Second, given the new equilibrium prices, it becomes relatively cheaper to opt for automated patching at a discount of $p^*(1-\delta^*)$. This provides additional incentives to encourage better security behaviors. On the other hand, a higher price can be detrimental to usage and associated revenues, and a reduced unpatched population can create incentives for users who were patching under the status quo to now remain unpatched.

The net impact of these effects depends on which consumer market structure is induced by the vendor's new prices. As Lemma 5 demonstrates, the vendor may induce a segmentation characterization with threshold orderings of either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$. For the latter structure which matches the status quo, the threshold v_b increases and threshold v_a decreases relative to the status quo in equilibrium under PPR. Thus, the size of the unpatched population (i.e., $u = v_a - v_b$) shrinks as it is compressed on both ends. However, the threshold v_p increases because of the patching rights premium. In aggregate, the vendor is able to increase profitability by charging a premium to high-tier consumers (valuations in $[v_p, 1]$) who are willing to pay the premium to protect from valuation-dependent losses and to low tier consumers (valuations in $[v_b, v_a]$) who are willing to pay the premium because of the smaller security externality that is associated with a smaller equilibrium unpatched user population.

For the other threshold ordering that takes the form $0 < v_a < v_b < v_p < 1$, there is a restructuring in the consumer market segments (see the discussion following Lemma 5). It is in the vendor's best interest to have a relatively large patching rights premium in this region which makes the retaining of patching rights only incentive compatible for the middle and higher-valuation users (valuations in $[v_b, 1]$). Low-valuation users respond to a substantial discount by forgoing patching rights and switching to automated patching. Because low-valuation users tend to be the ones with reduced incentives to patch and protect themselves, the market segmentation that occurs also leads to a smaller unpatched population and less resultant security risk. In a similar spirit to the discussion above, this is profitable to the vendor as it is able to raise prices due to greater security and greater

willingness to pay to retain patching rights by middle and high-valuations users.

The relative improvement in profitability associated with PPR in (25) and (29) highlights the type of market characteristics where efforts for a vendor to reexamine patching rights is more fruitful. In particular, the relative improvement in profitability is increasing in the cost of automated patching (c_a) and decreasing in standard patching costs (c_p). As the cost of automated patching increases through the relevant region (see Proposition 2), under the status quo the vendor necessarily reduces the software's price to make the automated patching option continue to be affordable. This is important because it prevents a significant loss in users resulting from higher security risk that can arise if the automated patching option becomes too costly. On the other hand, under PPR the vendor can achieve a similar effect by strategically adjusting the discount targeted to the users of the automated patching option rather than the entire user population. With regard to standard patching costs, when they decrease the vendor achieves a relatively larger increase in profits. In this case, the premium charged to users who elect to retain patching rights can be increased because these costs are lower.

Section 4.1 establishes that high independent risk $(\pi_i \alpha_i)$ precludes a segment of unpatched users from forming in equilibrium. Provided that the level of risk satisfies an explicit lower bound, small changes in risk cannot affect profitability. However, when independent risk is at a low to moderate level, it can impact the profitability of a PPR policy.

Corollary 1 *There exists a bound $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $\pi_i \alpha_i < \min \left[\frac{c_p \pi_a \alpha_a}{1 + c_p - c_a}, \frac{c_p}{1 + c_p} \right]$, then the increase in profitability associated with PPR decreases in α_i.*

Corollary 1 shows that the profitability of PPR tends to decrease in unpatched risk that is independent of the size of the unpatched population. Said differently, the value of a PPR policy is higher for the vendor when users have lower inherent incentives to patch. In today's computing environment, the most commonly exploited vulnerabilities are ones with patches available, some having been available for years. This observed, persistent unpatched usage of software is strongly suggestive that $\pi_i \alpha_i$ itself must be limited in magnitude; this is a necessary condition for unpatched usage to exist. These observations coupled with Corollary 1 imply that real-world parameter sets tend to be on a portion of the space where a PPR policy has relatively increased profitability.

Figure 2 illustrates how the value of PPR is affected under varying security-loss environments. The percentage increase in profitability is plotted under three parameter sets. Case A represents a baseline case with moderate standard patching costs $(c_p = 0.6)$ and automated patching instability
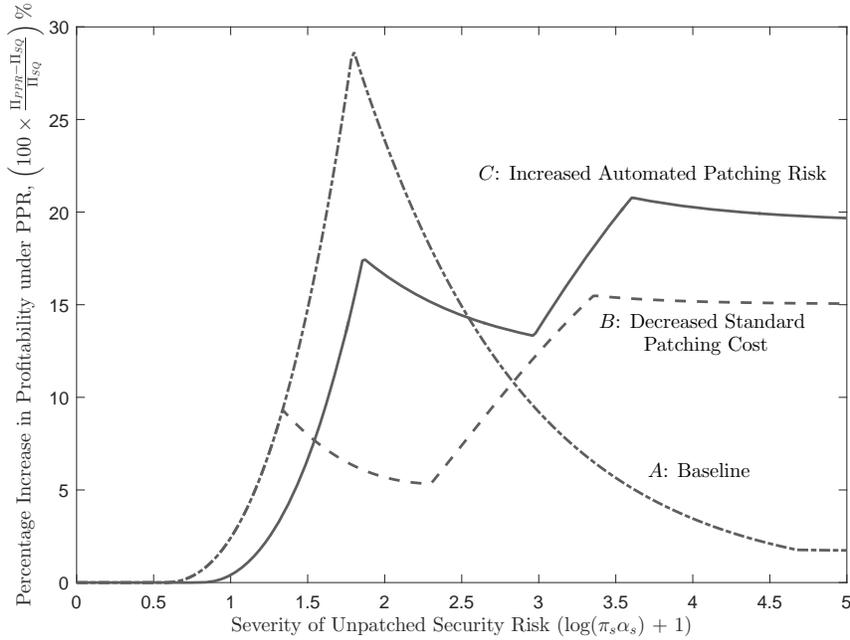
Figure 2: Percentage increase in profitability under priced patching rights: the impact of changes in automated patching instability losses and standard patching costs. Case A illustrates the baseline which employs moderate patching costs ($c_p = 0.6$) and automated patching risk ($\pi_a \alpha_a = 0.55$). Case B decreases patching costs slightly ($c_p = 0.5$), whereas Case C increases automated patching risk slightly ($\pi_a \alpha_a = 0.65$). Independent security risk ($\pi_i \alpha_i = 0.15$) and automated patching costs ($c_a = 0.1$) are selected to ensure unpatched and automated patching behaviors are present.

risk ($\pi_a \alpha_a = 0.55$). In Case B, standard patching costs are decreased slightly ($c_p = 0.5$) while automated patching instability risk ($\pi_a \alpha_a = 0.55$) is held constant. Similarly, in Case C, automated patching instability risk is increased slightly ($\pi_a \alpha_a = 0.65$) while standard patching costs are held constant ($c_p = 0.6$). Both status quo pricing and PPR induce the same consumer market structure characterization with a threshold ordering of $0 < v_b < v_a < v_p < 1$ under sufficiently high dependent risk (i.e., the right-hand side of the figure starting near 3.5 on the x-axis) for all curves. Proposition 2 establishes that, under these market characteristics, the percentage increase in profitability decreases in standard patching costs (c_p); this can be seen by comparing curve A to B in the right-hand side of the figure. On the other hand, the relative profit improvement increases in automated patching instability risk ($\pi_a \alpha_a$) which can be observed by similarly comparing curve A to C in the same region. Figure 2 demonstrates that a PPR policy can significantly boost profits even under relatively moderate security risk.

Section 4.1 examines how a vendor's patch release frequency impacts the value of PPR in terms of relative profitability for the vendor. Some of the insights from that discussion carry over here

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despite some consumers remaining unpatched in equilibrium even under high effective security risk. In addition, the following corollary establishes how a vendor's patch release frequency impacts risk associated with unpatched usage.

Corollary 2 *There exists a bound $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $\pi_i \alpha_i < \min\left[\frac{c_p \pi_a \alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$ and $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$, then the reduction in the size of the unpatched population, $u_{SQ}^* - u_P^*$, is decreasing in c_p and increasing in $\pi_a \alpha_a$.*

For software vendors whose market outcomes currently have all segments represented under status quo pricing, the impact on security would be greater for software with bundled patch releases than for more frequent patch releases. Software with bundled patch releases have higher expected automated patching losses that push low-valuation consumers toward unpatched usage. In this sense, vendors who currently bundle their patches instead of using a frequent patch release strategy have the most to gain in terms of improving software security through a PPR policy.

Another interesting implication of our model concerns a comparison of prices under the status quo and under optimal PPR. As Eckalbar (2010) demonstrates, the bundled price is higher for mixed bundling than for pure bundling. Thus, one might expect that if p_{SQ}^* is the price under the status quo, then $\delta^* p^* < p_{SQ}^* < p^*$ is satisfied in equilibrium when patching rights are priced. That is, users who want to retain patching rights pay a premium and users who opt for automated patching receive a discount relative to the status quo. However, the following proposition demonstrates that the vendor may strategically raise both prices in equilibrium, in comparison to status quo pricing.

Proposition 3 *There exists a bound $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1-c_p)\sqrt{1-\pi_a \alpha_a}$ and $\pi_i \alpha_i < \min\left[\frac{c_p \pi_a \alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$, when either*

(i) $c_a < \min\left[\pi_a \alpha_a - c_p, c_p(1-\pi_a \alpha_a), \frac{\pi_i \alpha_i (2\pi_a \alpha_a - 1) - 1}{2\pi_a \alpha_a + \pi_i \alpha_i - 3} - \pi_a \alpha_a\right]$, or

(ii) $|\pi_a \alpha_a - c_p| < c_a < \min\left[c_p(1-\pi_a \alpha_a), \frac{(1-\pi_a \alpha_a)(\pi_i \alpha_i - 2c_p + 1)}{-4\pi_a \alpha_a - \pi_i \alpha_i + 5}\right]$,

the vendor prices patching rights such that both p^* and $\delta^* p^*$ are higher than the common price, p_{SQ}^*, when patching rights are endowed to all users.

Not only does the endowment of patching rights lead to excessive security risk due to poor patching behavior, it also fails to reflect the value of security provision being offered by the vendor. Vendors who create better, more secure solutions for their customers should be able to harvest some of that value creation via increased prices. Proposition 3 highlights this important point by characterizing

broad regions where the vendor increases the price of both options above the single price offered in the case of the status quo. This occurs for a lower level of automated patching costs (c_a), and the reason both prices increase is twofold. First, users who prefer to retain patching rights are willing to pay more for smaller unpatched populations (i.e., reduced security risk) and control over their own patching process. Second, the value associated with cost-efficient and more secure, automated patching options is more readily harvested when users of this option are ungrouped from users who choose not to patch under the status quo. A PPR policy helps to enable this separation. Thus, when a vendor differentiates in this manner based on "rights," he can simultaneously increase prices, encourage more secure behaviors, and generate higher profits. The outcome under this business strategy is noteworthy because it is starkly different than one in which security protections are sold and those who opt out are both unprotected and cause a larger security externality.

Proposition 3 suggests that usage may become more restricted under PPR. Moreover, it is unclear how specific costs associated with security would be affected as consumers strategically adapt their usage and protection decisions. Proposition 2 demonstrates that PPR can reduce the size of the unpatched population relative to the status quo, which in turn implies the risk associated with security attacks decreases. However, the magnitude of losses associated with these attacks critically depends on who actually bears them as they are valuation-dependent and consumers' equilibrium strategies will shift when patching rights are priced. The expected losses associated with security attacks stemming from the unpatched population $u(\sigma^*)$ can be expressed

$$SL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP)\}} \left(\pi_s \alpha_s u(\sigma^*) + \pi_i \alpha_i \right) v dv \,. \tag{31}$$

In a similar fashion, the expected losses associated with configuration and instability of automated patching are denoted by

$$AL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,AP)\}} c_a + \pi_a \alpha_a v dv \,, \tag{32}$$

and the total costs associated with standard patching by

$$PL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)\}} c_p dv \,. \tag{33}$$

The net impact of consumers changing their patching strategies (standard patching, remaining unpatched, and electing for automated patching) on these security-related costs is unclear. In order to examine these concerns in aggregate, it is useful to define total security-related costs as

the sum of these three components:

$$L \triangleq SL + AL + PL \, , \tag{34}$$

in which case social welfare can be expressed as

$$W \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v) \in \{(B,NP),(B,AP),(B,P)\}\}} v dv - L. \tag{35}$$

The following proposition establishes that when automated patching costs are not too large, PPR can in totality have a negative effect on social welfare. This result is interesting in that both losses associated with security attacks and total costs associated with standard patching can be shown to decrease when patching rights are priced, and yet PPR can still be detrimental from a welfare perspective.

Proposition 4 *There exists a bound $\tilde{\alpha}_s$ such that when $\alpha_s > \tilde{\alpha}_s$, if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$ and $\pi_i \alpha_i < \min \left[\frac{c_p \pi_a \alpha_a}{1 + c_p - c_a}, \frac{c_p}{1 + c_p} \right]$, then PPR can either decrease or increase security attack losses, but leads to a small decrease in social welfare. Technically, $PL_P < PL_{SQ}$, $AL_P > AL_{SQ}$, $W_P < W_{SQ}$ and*

(i) if $c_a < \min \left[\pi_a \alpha_a - c_p, c_p(1 - \pi_a \alpha_a) \right]$, and

$$\frac{4c_p^2 \pi_a \alpha_a}{-c_a + c_p + \pi_a \alpha_a} - \frac{(c_a(2 - \pi_a \alpha_a) + \pi_a \alpha_a(1 - \pi_a \alpha_a))^2}{(1 + c_a - \pi_a \alpha_a)(1 - \pi_a \alpha_a)} > 0 \, , \tag{36}$$

then $SL_P > SL_{SQ}$;

(ii) otherwise, $SL_P \le SL_{SQ}$.

The parameter region in Proposition 4 corresponds to automated patching costs (c_a) being relatively lower and satisfying the conditions of Lemmas 4 (first part) and 5. Recalling that under PPR, the consumer market structure can be characterized by automated patchers being either in the middle tier (i.e., $0 < v_b < v_a < v_p < 1$) or in the lower tier (i.e., $0 < v_a < v_b < v_p < 1$) in equilibrium, first consider the former case where the consumer market structure matches the characterization under the status quo. By pricing patching rights, the vendor will induce an expansion of the consumer segment that elects for automated patching on both sides. That is, some unpatched users as well as some standard patching users under the status quo will now choose automated patching.

Additionally, some unpatched users are now out of the market due to the increase in the price p^* associated with retained patching rights (technically, the threshold v_b increases). Therefore, losses associated with unpatched security attacks and costs associated with standard patching are both lower in comparison to the status quo, i.e., $SL_P < SL_{SQ}$ and $PL_P < PL_{SQ}$.

However, expansion of the consumer segment choosing automated patching turns out to be costly. In particular, because consumers have the opportunity to relinquish patching rights to save the premium $(1-\delta^*)p^*$, those that make up the expansion of this segment may incur greater security investments and system instability losses in order to avoid paying this premium. For example, at the higher end of the valuation space, a consumer may have incurred only c_p under status quo pricing but when incentivized to shift to automated patching because of the discount, she now incurs a security cost of $c_a + \pi_a \alpha_a v$ which is valuation-dependent and can exceed c_p. A similar increase in costs can arise at the lower end as consumers shift from losses associated with security attacks to investments and instability losses associated with automated patching. Proposition 4 establishes that the decrease in usage and increased aggregate costs incurred related to automated patching ultimately outweigh the reduction in security attack losses and standard patching costs from a welfare perspective. From the software vendor's perspective, the ability to market product offerings as geared to reduce security risk and attack losses while increasing profits is enticing, and having awareness of the impact on welfare can help shape these initiatives.

When the vendor's pricing behavior induces a restructuring of segmentation in the consumer market (i.e., $0 < v_a < v_b < v_p < 1$), the outcome is similar but has some nuanced differences. In this case, consumers whose equilibrium strategy is to retain patching rights but not patch (users with valuations between v_b and v_p) have higher valuations than those preferring this strategy under the status quo case. Thus, even though the size of the unpatched population, $u(\sigma^*)$, decreases under PPR, the higher valuations of the consumers exhibiting the risky, unpatched behavior can result in them incurring higher losses when bearing security attacks. It hinges on whether $u(\sigma^*)$ decreases sufficiently to offset the higher valuations of the risky population. In part (i) of Proposition 4, the conditions required for the restructured consumer market as laid out in Lemma 5 appear. Further, (36) provides the condition whereupon security attack losses are, in fact, higher under PPR, despite the reduction in unpatched usage. One can think of this outcome as characterized by fewer attacks but on higher value targets leading to greater losses in equilibrium. This condition tends to be satisfied as the likelihood of automated patch instability increases, which provides more incentive for consumers to remain unpatched instead. With the potential of security attack losses to also

increase, welfare is even further suppressed compared to the status quo.

Encouragingly, there are several regions where social welfare is positively impacted by PPR as well. One example is when automated patching costs are at a level large enough that an automated patching segment is absent under the status quo but small enough that this segment arises when patching rights are priced (see the second part of Lemma 4 and Lemma 5).

Proposition 5 *There exists a bound $\tilde{\alpha}_s$ such that, when $\alpha_s > \tilde{\alpha}_s$, if $1-\pi_a\alpha_a-(1-c_p)\sqrt{1-\pi_a\alpha_a} < c_a$ $< c_p(1-\pi_a\alpha_a)$ and $\pi_i\alpha_i < \min\left[\frac{c_p\pi_a\alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$, then PPR leads to decreased security attack losses and an increase in social welfare. Technically, $SL_P < SL_{SQ}$, $PL_P < PL_{SQ}$, $AL_P > AL_{SQ}$, and $W_P > W_{SQ}$.*

Proposition 5 examines a higher cost of automated patching in which case, under status quo pricing, the consumer market equilibrium is characterized by an absence of automated patching (i.e., the threshold ordering of $0 < v_b < v_p < 1$ in Lemma 4). One can think of this as a context where automated patching technology is somewhat inferior and users elect not to use it in equilibrium. This behavior can result in a large unpatched population and substantial security risk, causing many potential consumers to prefer not to be users of the product. Thus, the value of a PPR policy can be lucrative if it provides sufficient incentives to reduce unpatched behavior and expand usage. Under an optimally set PPR policy, users who were unpatched under the status quo are incentivized by a discount to use the automated patching option. In that automated patching is an inferior technology in this context, these users may bear greater costs and instability losses associated with automated patching in exchange for receiving this discount. These greater costs are detrimental to welfare.

On the other hand, because the unpatched population is significantly reduced, losses associated with security attacks are lower ($SL_P < SL_{SQ}$). Moreover, because the vendor makes the automated patching available at a discount, usage in the market for the software expands. In fact, when the loss factor on automated patching technology ($\pi_a\alpha_a$) is at the high end of the focal region, both the price of the product with patching rights (p^*) and without (δ^*p^*) can be lower than the price under the status quo (p_{SQ}^*). Thus, usage in the market can expand substantially, and the additional surplus generated from these consumers who were non-users under the status quo helps to benefit welfare. Proposition 5 establishes that the net effect of these factors is positive, and PPR has a positive influence on social welfare.

While a PPR policy is quite effective at reducing unpatched populations and losses associated
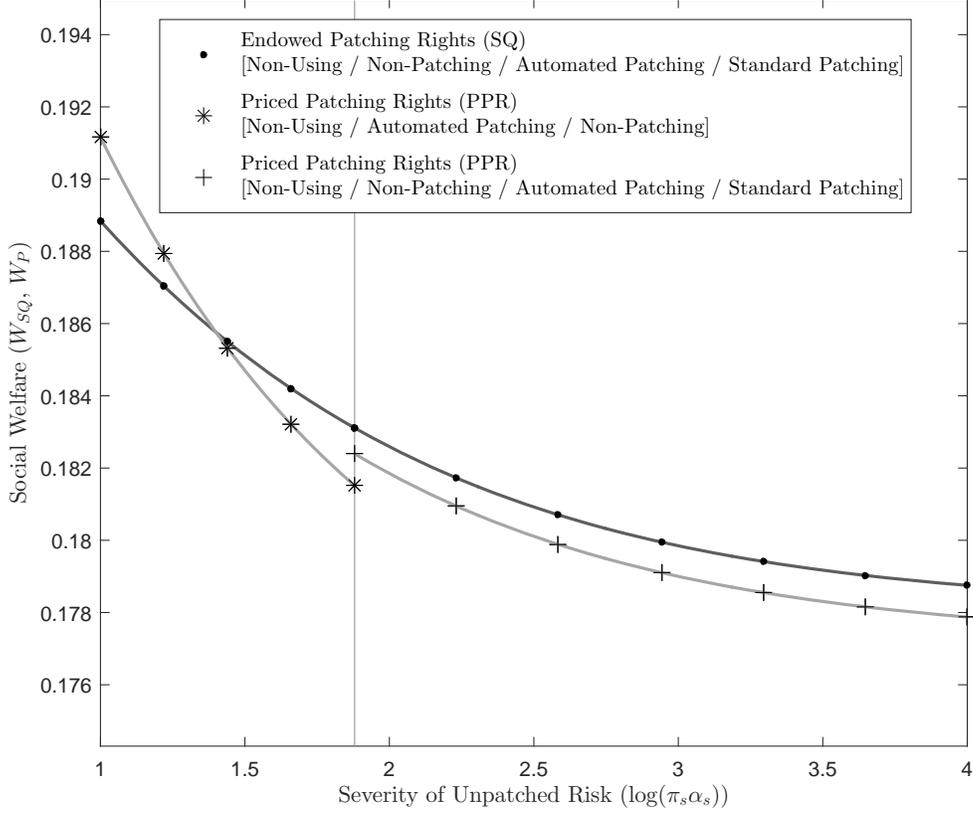
Figure 3: Beneficial impact of priced patching rights on social welfare as unpatched security risk becomes lower. The parameter values ($c_a = 0.1$, $\alpha_i = 0.2$, $\pi_i = 0.1$, $\alpha_a = 3.5$, $\pi_a = 0.1$, and $c_p = 0.4$) are chosen so that all consumer segments are represented in equilibrium when patching rights are endowed. In such cases, social welfare has been shown to be relatively depressed under higher unpatched security risk which is consistent with the right-hand side of the figure. The consumer segments that arise in each region are presented in order of increasing consumer valuations (from left to right) in the legend.

with security attacks, Propositions 4 and 5 demonstrate that its impact of welfare can be mixed when security risk is large. Focusing on relatively smaller automated patching costs as in Proposition 4, Figure 3 further illustrates how welfare is impacted as the security loss factor becomes smaller. As can be seen in the left-hand portion of the figure, a PPR policy can also be beneficial to social welfare relative to the status quo strategy as unpatched security risk ($\pi_s \alpha_s$) decreases. Under the status quo, the consumer market equilibrium is characterized by the threshold ordering $0 < v_b < v_a < v_p < 1$ throughout the plot. However, two distinct consumer market structures are represented under PPR. To the right of the discontinuity, the characterization of thresholds remains consistent with the status quo, while to the left of the discontinuity (hence lower $\pi_s \alpha_s$), the

threshold ordering becomes $0 < v_a < v_b < 1$. In other words, as unpatched security risk decreases, patching rights are priced in a way that significantly restructures the equilibrium consumer strategies in comparison to the status quo: consumers with high valuations retain patching rights but choose to remain unpatched, and consumers with lower valuations forgo patching rights and either shift to automated patching or exit the market.

What is most interesting about this reshuffling is that the consumers who were causing the security risk no longer do so and, as a result, the consumers who were incurring standard patching costs to shield themselves from the security risk also no longer need to do so. Organizations have long carried a large financial burden associated with the rigorous patching processes that they are forced to employ to limit risk. If the ecosystem becomes safer, these organizations could reduce these investments while keeping that risk exposure limited. This is the generalizable insight brought to light here – a PPR policy not only reduces security risk, it enables high-valuation users to avoid incurring typically large patching costs. The net result of PPR is that total costs related to automated patching increase (the automated patching population expands), costs associated with standard patching disappear (patching burden is relieved), and security attack losses stemming from unpatched usage is reduced (significant reduction in the size of the unpatched population). As a result, social welfare can increase under PPR in comparison to the status quo as the security loss factor decreases out of region covered by Proposition 4.

Figure 4 illustrates the finding from Proposition 5 that social welfare increases under PPR for a high security loss factor when the costs of automated patching are slightly elevated. This can be seen in the right-hand portion of the figure. Moreover, Figure 4 numerically demonstrates that the benefits to welfare extend to a much broader range of security losses. In summary, a PPR policy presents an opportunity for vendors of proprietary software to not only improve profits, but also to improve welfare by decreasing the magnitude of the externality generated by unpatched usage, even to the degree that the patching burden can be relieved.

5   Discussion and Concluding Remarks

In the current state of affairs, both software end users and system administrators are faced with a barrage of security patches. However, because users are endowed with the right to choose whether or not to apply these security updates, a large portion of the user base ultimately chooses to remain unpatched, leaving their systems prone to security attacks. These users contribute to a security
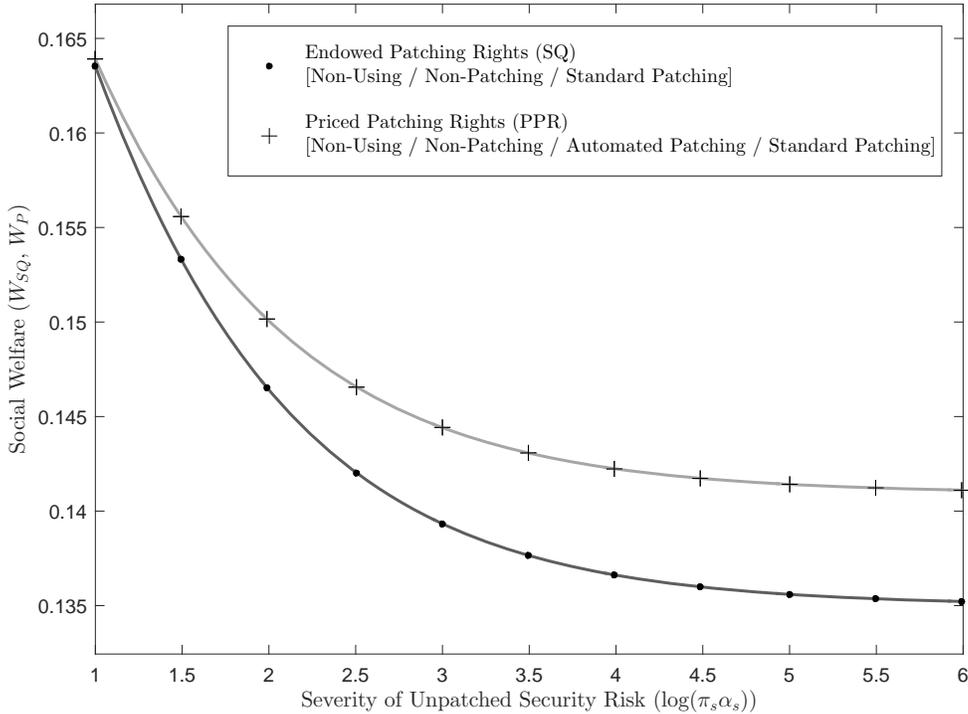
Figure 4: Beneficial impact of priced patching rights on social welfare when automated patching costs are elevated. The parameter values ($c_a = 0.2$, $\alpha_i = 0.2$, $\pi_i = 0.1$, $\alpha_a = 3.5$, $\pi_a = 0.1$, and $c_p = 0.4$) are chosen so that the relative costs of automated patching are high enough that no consumer prefers this option when patching rights are endowed but not so high that automated patching technology is generally prohibitive. In such cases, a PPR policy can engender its use which is efficient. The consumer segments that arise in each region are presented in order of increasing consumer valuations (from left to right) in the legend.

externality that affects all users of the software, which degrades its value and has a negative impact on its profitability. We study an adapted business model where a software vendor also differentiates its product based on patching rights. In this model, the right to choose whether or not to patch is no longer endowed. Instead, consumers who prefer to retain these rights and hence control of the patching status of their systems must pay a relative premium. Consumers who prefer to relinquish these rights have their software automatically updated by the vendor and, in exchange, end up paying a relatively discounted price in equilibrium. The market segmentation induced carries a reduction in security risk and an increase in profitability to the vendor. In this way, a PPR policy can be a beneficial marketing strategy driving revenue growth; a vendor can market its product offerings as being more secure because its differentiated products incentivize better security behaviors by users.

A PPR policy is a very effective way to improve software security. The size of unpatched users decreases in equilibrium under PPR which, in turn, tends to lead to a decrease in losses associated with security attacks on systems running the software. In some cases, the firm may raise both the price of the premium version with patching rights and the price of the "discounted" version without patching rights relative to the optimal price offered under the status quo. This demonstrates how a software firm can extract value from a combination of improved automated patching technology and a pricing strategy that incentivizes better security outcomes that are valued by consumers. A PPR policy can negatively affect social welfare when usage in the market significantly contracts, but in many cases it increases welfare as a result of the lower resulting security risk.

Our study is a simplification of a software vendor's versioning strategy which, more practically, can involve managing numerous quality-differentiated versions, bundled security/feature updates, interim update versus major release cycles, and planned obsolescence. The intent of our simplification is to put a spotlight on the value of PPR as a tool that provides incentives for lower-valuation users to engage in better behaviors that help reduce the effective security externality. Even in the complex settings that exist in the software industry, the ideas and insights stemming from our work can improve software outcomes if, however implemented, lower-valuation users ultimately select versions that remain up to date with security patches. In the software industry, vendors have taken greater actions to ensure that versions of their products stay updated (e.g., the offering of automatic updates, configuring of default options to turn automatic updates on, and even versions of software which forcibly update). Our work helps clarify the impact of actions that aim to incentivize lower-valuation users to forgo patching rights.

Despite the benefits, there are many frictions associated with the essence of a PPR policy which can cause vendors to resist adoption. First, by forcibly updating users who choose to forgo patching rights for discounted versions of the software, the vendor is exposing these users to system instability risk. While these losses are internal to the model and part of the trade-off evaluated by users, vendors may shy away from additional exposure to liability. Second, whenever there is a patch that leads to severe instability, vendors may likely receive backlash from consumers and incur damage to their reputation. In this regard, any vendor strategy that eliminates patching rights from a market segment may go hand-in-hand with investments in patch quality. In that many software vendors have been providing the option of automatic updates for years now, they have made significant progress on patch stability. In order for more vendors to consider adoption of a PPR policy (or practical variants), assurance of patch stability will be critical because the policy

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leaves the targeted consumer tier with little recourse.

5.1 Alternative Models and Application Domains of PPR

The manner in which software is licensed and updated is constantly changing due to technology disruptions and the developing needs of consumers. While there is some diversity in observed licensing strategies in the software industry, the current work focuses on an important and large class of software products where patches are predominantly made freely available to users who then make decisions whether or not to install them. This class includes versions of software products from an extensive list of top vendors such as Microsoft, Oracle, Symantec, VMware, and IBM.[13] We discuss both alternative models to PPR and other potential application domains of PPR, highlighting in the latter whether PPR-type policies are likely to be effective in the domain of interest.

Some producers make software source code available for free and build revenue models around service and support (e.g., Red Hat Enterprise Linux, Elasticsearch, and Oracle Java). In this open-source domain, when software is made available for free, developers have no obligation to provide patches. One approach that has been observed is the offering of patches only to paying customers who have contracted for support. This alternative model involves charging for patches. Oracle has done this by offering the patches only for paying customers of its freely available open-source Java software (Krill 2015). We examine how this strategy compares to PPR. First, it is important to understand what happens when an OSS developer does not provide patches to its users. In such a setting, a consumer's options can include: (i) use the software for free and be unpatched, risking security attacks, (ii) not use the software, and (iii) leverage the open-source nature of the software to self-patch.[14] Because the software has zero price, if the effective security risk is low, *all* consumers will use the software and obtain positive surplus in equilibrium. On the other hand, if the effective security risk exceeds a certain bound, then users can only enter until the size of the unpatched population reaches a critical size upon which dependent risk wipes out all surplus derived from any unpatched use of the software.

In such a context, the offering of a patch alone will significantly improve welfare. Patches enable high-valuation users to shield themselves from risk, which in turn can reduce the security

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[13]Examples of specific software products that can be installed on-premises and maintained by users include Microsoft Windows, Microsoft Office, Microsoft SQL Server, Oracle Database, Oracle WebLogic Server, Norton Antivirus, Symantec Endpoint Protection, VMware Workstation, and IBM WebSphere.

[14]Only a select group of users might have the capability to self-patch.

externality that decays the surplus of users who choose to remain unpatched. Suppose a vendor offers patches only to those who agree to pay for them as part of a support package. This provides users the additional usage option to pay for patches. Examining conditions analogous to Section 4.2, the equilibrium consumer market structure will be characterized by both patched and unpatched usage. Because high-valuation consumers can now patch and protect themselves from attack, social surplus will increase in equilibrium in comparison to the case where it is mostly wiped out in the presence of high risk and patch unavailability.

One can compare our PPR policy to this alternative model where an OSS developer prices the patch itself. A PPR policy will tend to dominate this alternative model both in terms of profitability and in terms of welfare. It is straightforward to see why it is more profitable. Because a PPR policy charges premia for both unpatched and standard patched usage and even a positive price for automatic patched usage, it generates a lot of direct revenue. The alternative model requires all the revenues to come from higher-valuation users who need to be willing to pay for access to patches. By giving the software away for free, one's revenue model typically hinges on the creation of positive network effects and leveraging those to create higher willingness-to-pay within the higher-value segment. However, this alternative model has payment tied to security patches, which means that the higher value segment will only be willing to pay for patches if the security risk is large. This suggests that risk stemming from expanded usage is necessarily required, but that lies in stark contrast to any sort of positive network effects strategy. These opposing effects handicap this model in comparison to the PPR policy being advocated here.

In terms of social welfare, this alternative model suffers relative disadvantages dependent upon the region of concern. If automated costs are comparable to standard patching costs, a PPR strategy expands overall usage relative to the status quo by incentivizing more users to select automated patching. This usage expansion is also much larger than can be expected under the alternative model because in that case users have fewer options and cannot readily shield risk without paying a premium – this lies in contrast to the PPR strategy where unpatched users are provided incentives instead to reduce risk. If automated patching costs are large and it becomes difficult to incentivize automated patching, then the alternative model can in fact generate greater usage than under a PPR policy. However, expanded usage of the free product increases the externality to the point where unpatched users gain no surplus in the alternative model. Therefore, even in this case, the PPR policy achieves higher welfare with a smaller user segment, where they derive positive surplus from the reduced security risk.

Even for OSS providers who are not charging for patches, a PPR policy can be a valuable option. In particular, OSS providers who are already employing other monetization strategies can easily implement the essence of a PPR policy and benefit via improved security and revenues. On the other hand, OSS providers who are not currently differentiating their offerings and whose mission is perhaps more squarely centered on public good are not likely to be good candidates for PPR. Such providers may find both the concept of giving up rights and charging for software to be inconsistent with their mission.

Next, there are many producers who offer subscriptions for access to SaaS (e.g., Salesforce CRM, Workday, and NetSuite). In this domain, a vendor manages the patching process directly so that consumers no longer make a decision about whether or not to patch. However, this certainly comes at a cost to the provider, which ultimately gets reflected in the price consumers pay. SaaS software tends to run on a fairly limited set of servers under the control of the vendor, whereas in the on-premises model the number of systems running software tends to be much larger with decision-making rights being distributed. In the case of on-premises software studied in our model, it is often the case that patchable vulnerabilities are exploited. This occurs because malicious individuals study the vulnerability that has been addressed in the patch, and then exploit the same vulnerability hoping to successfully attack some subset of users who remain in an unpatched state. In the case of SaaS, the approach to finding vulnerabilities differs significantly as both the software as well as patches to the software remain internal to the vendor. Malicious agents must find vulnerabilities in the interface to the SaaS product itself. On the consumer side, it can be the case that companies (particularly small and medium-sized businesses) fully commit to the cloud as part of their IT strategies, which may introduce correlated risks on cloud platforms and make it difficult to analyze a PPR strategy in an context where a vendor additionally offers both on-premises and SaaS alternatives. Because of the many differences in characteristics that surface when comparing SaaS to on-premises software, a PPR policy does not fit the SaaS domain very well. However, studying the role of PPR under the additional complexity of mixed offerings (SaaS and on-premises) is potentially a fruitful direction for future research because of the distinct security risks associated with the different licensing models (August et al. 2014)

the service. Comparing this model to pricing patching rights, we find that when the service quality is high enough, then the SaaS model outperforms pricing patching rights both in terms of profitability and in terms of welfare. This is clear since consumers incur no patching losses to be shielded from attack, which then allows the vendor to extract more surplus and be more profitable.

On the other extreme, when γ is low enough, then the software is of poor quality so that the SaaS model is outperformed by pricing patching rights. When γ is in an intermediate range, then the SaaS model outperforms pricing patching rights in terms of welfare while pricing patching rights is more profitable. Under high security risk, the equilibrium market size under a priced patching rights strategy in which all market segments are present (either $0 < v_b < v_a < v_p < 1$ or $0 < v_a < v_b < v_p < 1$) would be larger than the total market size under the SaaS model. However, consumers incur losses through either patching or bearing security risks that ultimately make welfare worse off in aggregate.

Software producers have long shielded themselves from liability using well-crafted license agreements. However, the increasing breadth of software use in riskier operating environments (including the critical infrastructure, biomedical products, and automobiles) comes with increased exposure to strict liability. In particular, liability concerns come to the forefront in software contexts where system failure can lead to safety or health hazards. For example, researchers have demonstrated the ability to hack into automotive systems and hijack control over brakes and steering from drivers; vulnerabilities such as these can be utilized to cause physical harm to citizens (Greenberg 2015). In application domains with this property, a PPR policy may be inappropriate. In our model, these settings would be characterized as having high automated patching risk (i.e., a large $\pi_a \alpha_a$ parameter), in which case a PPR strategy is shown to be weakly dominated. Intuitively, manufacturers would likely not agree to have embedded software on products such as pacemakers automatically updated. In the presence of strict liability, assurance of quality takes precedence.

Many issues tackled in this paper will become increasingly salient as the Internet of things (IoT) comes of age. With IoT, the explosion of interconnected devices will be accompanied by both increased vulnerabilities and increased threats as malicious actors evolve to exploit new possibilities. The standard patching processes employed by organizations will face significant challenges with the scale of device growth and their complex interactions. IoT's scale will require a greater level of automation in patch management. The extent and type of human interaction taking place with software that drives servers, laptops, tablets and phones is fundamentally different than that with the embedded software that drives IoT devices. This, in turn, may necessitate greater sophistication with the automated management of IoT devices. In this landscape, IoT presents as both a challenge and opportunity for security interventions like a PPR policy. For PPR to be effective in this domain, automated patching technology must first improve and attain a service level where minimal failure rates are observed. Until then, it may be premature to consider a PPR policy for IoT devices.

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5.2   Concluding Remarks

Standard patching costs (c_p) may increase with factors such as the relative size of a user firm, for instance, depending on the number of servers on which they run the software product. A primary driver is that a larger number of users require a higher number of servers likely in different environments and configurations on which to patch the software, driving up the patching costs proportional to the number of servers. Our results however continue to hold with consideration of this dimension: In our model, each host can be considered to be a single server and a large corporation can be thought to have a number of different servers. Our analysis is unaffected provided each decision maker owns at most countably many hosts. It may also be the case that two organizations conducting the same tasks associated with a rigorous, standard patching process would have some variation in the costs incurred. For example, there may be some variation around one thousand dollars per server. Having some variation in this regard will not qualitatively change any of our results for two reasons. First, given the nature of the tasks being performed, there will always be a primary, valuation-independent component of these costs which is currently captured. Second, inclusion of a valuation-dependent component will ultimately be swamped by the valuation-dependent losses that are currently present in the model. In particular, in our model, both losses due to system instability and losses due to security attacks are valuation-dependent. Importantly, both of these losses are much more strongly associated with valuation than the residual component of patching costs, which limits its impact.

We focus on the case where a software vendor releases and immediately applies automatic updates to all systems that have opted in to this form of patching. One interesting extension would be to explore the dynamics of automated patch deployment. In particular, some vendors may prefer to slowly roll out an automated update to subsets of systems that have opted-in and subsequently ramp up roll out as the vendor builds confidence in the quality of its patch (as measured by monitoring of systems that have been updated). Another related extension would be to explore how a vendor's investment in patch quality interacts with this dynamic deployment strategy.

Our study constructs a model of a monopolist software vendor to study PPR. In practice, essentially no sector is monopolistic and there is some competition in any industry and market. However, as long as the firms in the market have some market power (which is true in almost every industry and market, and certainly for the software industry), a monopolistic model captures most of the insights that would come out from the intended arguments in an oligopoly model of the same

situation in a much clearer and transparent way. In general, increased competition will negatively affect firm profits and likewise the profitability of PPR, and studying competition-related questions would be an interesting direction for future research.

Under the framework of a PPR policy, the consumers who pay a premium to retain patching rights need no additional monitoring by the proprietary vendor or OSS provider to examine patching status. However, consumers who benefit financially in exchange for these rights must have their systems automatically updated per the contract. This requires a careful implementation that entails the monitoring of systems. First, the updating of systems need not be in full control by the vendor nor instantaneous to derive the benefits of this policy. For example, consumers can be given a time window to apply patches before they are forcefully installed. This gives users some leeway operationally. Second, vendors can modify the implementation of PPR to the environment in which they reside. For example, mobile application developers could give a discount to users willing to automatically update their software, and even give a slightly greater discount to those who are willing to receive those updates immediately through their mobile data plan. Lastly, a user can always disconnect her system from the Internet to avoid the deployment of automated security updates. In this case, although the patches are not installed, the externality imposed by this unpatched system would also be partially reduced.

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Online Supplement for
Market Segmentation and Software Security:
Pricing Patching Rights

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Proofs of Propositions

A   High $\pi_i\alpha_i$

1   Status Quo

Lemma A.1 *When $\pi_i\alpha_i > 1$, under the status quo, i.e., $\delta = 1$, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_a < 1)$, *where* $v_a = \frac{p+c_a}{1-\pi_a\alpha_a}$:

   (A) $p + c_a + \pi_a\alpha_a < 1$

   (B) $c_p \geq c_a + \pi_a\alpha_a$

   (C) $c_p \geq \frac{c_a+p\pi_a\alpha_a}{1-\pi_a\alpha_a}$

(II) $(0 < v_a < v_p < 1)$, *where* $v_a = \frac{p+c_a}{1-\pi_a\alpha_a}$ *and* $v_p = \frac{c_p-c_a}{\pi_a\alpha_a}$:

   (A) $c_p < c_a + \pi_a\alpha_a$

   (B) $c_p > \frac{c_a+p\pi_a\alpha_a}{1-\pi_a\alpha_a}$

(III) $(0 < v_p < 1)$, *where* $v_p = p + c_p$:

   (A) $c_p + p < 1$

   (B) $c_p \leq \frac{c_a+p\pi_a\alpha_a}{1-\pi_a\alpha_a}$

Proof of Lemma A.1:   This is a sub-case in the proof of Lemma A.2, by setting $\delta = 1$. $\square$

Proof of Lemma 2:   We prove that if $\pi_i\alpha_i > 1$ and $\delta = 1$ (i.e., when patching rights are not priced), then if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$, we have that

$$p^* = \frac{1 - \pi_a\alpha_a - c_a}{2}, \tag{A.1}$$

<div align="center">

A.1

</div>

and σ^* is characterized by $0 < v_a < v_p < 1$ such that the lower tier of users prefers automated patching. On the other hand, if $c_a \geq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then

$$p^* = \frac{1 - c_p}{2}, \tag{A.2}$$

and σ^* is characterized by $0 < v_p < 1$ such that there is no user of automated patching in equilibrium.

Suppose $0 < v_a < 1$ is induced. Then the profit function is $\Pi_I(p) = p(1 - v_a)$. Using Lemma A.1, we have that $v_a = \frac{p + c_a}{1 - \pi_a \alpha_a}$. The optimal price is found to be $p_1^* = \frac{1}{2}(1 - c_a - \pi_a \alpha_a)$ with the corresponding profit $\Pi_I^* = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}$.

Similarly, suppose instead that $0 < v_a < v_p < 1$ is induced. Then the profit function is $\Pi_{II}(p) = p(1 - v_a)$. Using Lemma A.1, we have that $v_a = \frac{p + c_a}{1 - \pi_a \alpha_a}$. Again, the optimal price is found to be $p_2^* = \frac{1}{2}(1 - c_a - \pi_a \alpha_a)$ with the corresponding profit $\Pi_{II}^* = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}$.

Lastly, suppose that $0 < v_p < 1$ is induced. Then the profit function is $\Pi_{III}(p) = p(1 - v_p)$. Using Lemma A.1, we have that $v_p = c_p + p$. Now, the optimal price is found to be $p_3^* = \frac{1 - c_p}{2}$ with the corresponding profit $\Pi_{III}^* = \frac{(1 - c_p)^2}{4}$.

We next find conditions under which the maximizing price for each case indeed induces that market structure. For $0 < v_a < 1$, we need the set of conditions for Case (I) in Lemma A.1 to hold for p_1^*. To satisfy the first condition, we need $p + c_a + \pi_a \alpha_a < 1$ for $p = p_1^* = \frac{1}{2}(1 - c_a - \pi_a \alpha_a)$. This simplifies to $c_a + \pi_a \alpha_a < 1$, which is a preliminary model assumption. Then we need $c_p \geq \frac{c_a + p\pi_a \alpha_a}{1 - \pi_a \alpha_a}$ to hold for $p = p_1^*$ as well, which simplifies to $c_a \leq \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$. We also needed the condition $c_a \leq c_p - \pi_a \alpha_a$ for this case to hold. Since $\frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a} > c_p - \pi_a \alpha_a$ follows from $0 < c_p < 1$ and $0 < \pi_a \alpha_a < 1$, then the condition under which p_1^* would induce $0 < v_a < 1$ is $c_a \leq c_p - \pi_a \alpha_a$.

Similarly, for Case (II), the condition under which p_2^* would induce $0 < v_a < v_p < 1$ is $c_p - \pi_a \alpha_a < c_a < \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$. And lastly, for Case (III), the condition under which p_3^* would induce $0 < v_p < 1$ is $c_a \geq c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a$. Note that $c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a < \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$, so that when $c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a < c_a < \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$, we'll need to compare Π_{II}^* and Π_{III}^*.

Next, we find the conditions under which the maximal profits of each case dominate the other cases. In particular, when $c_a < c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a$, then $\Pi_I^* = \Pi_{II}^* > \Pi_{III}^*$. Since $c_p - \pi_a \alpha_a < c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a$ from $c_p < 1$, this implies that $0 < v_a < 1$ will be the resulting consumer market structure for $c_a \leq c_p - \pi_a \alpha_a$. Also, $0 < v_a < v_p < 1$ will be the resulting consumer market structure for $c_p - \pi_a \alpha_a < c_a < c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a$.

Also, for $c_a > \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$, we have that $\Pi_{III}^* > \Pi_{II}^*$, so that $0 < v_p < 1$ will be the resulting market structure when $c_a > \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$.

In between $c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a < c_a < \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$, we find the conditions under which Π_{II}^* dominates Π_{III}^*. Comparing the profits, we find that $\Pi_{II}^* \geq \Pi_{III}^*$ when $c_a \leq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{(1 - \pi_a \alpha_a)}$. Then we note that $c_p - \frac{1}{2}(1 + c_p)\pi_a \alpha_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{(1 - \pi_a \alpha_a)} < \frac{(2c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{2 - \pi_a \alpha_a}$ always holds, so that the resulting market structure when $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$ is $0 < v_a < v_p < 1$, and the resulting market structure when $c_a \geq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$ is $0 < v_p < 1$. $\square$

2 Pricing Patching Rights

Lemma A.2 *When $\pi_i \alpha_i > 1$, under PPR, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_a < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$:

    (A) $\delta p + c_a + \pi_a \alpha_a < 1$

    (B) $c_p + (1 - \delta)p \geq c_a + \pi_a \alpha_a$

    (C) $c_p \geq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

(II) $(0 < v_a < v_p < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ *and* $v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}$:

    (A) $c_p + (1 - \delta)p < c_a + \pi_a \alpha_a$

    (B) $c_p > \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

(III) $(0 < v_p < 1)$, *where* $v_p = p + c_p$:

    (A) $c_p + p < 1$

    (B) $c_p \leq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$

Proof of Lemma A.2: First, we establish the general threshold-type equilibrium structure. The proof of this is a sub-case of the argument in Lemma A.4, with the size of the unpatched user population $u = 0$ since $\pi_i \alpha_i > 1$. This establishes the threshold-type consumer market equilibrium structure.

Next, we characterize in more detail each outcome that can arise in equilibrium, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose the automated patching option, i.e., $0 < v_a < 1$, based on the threshold-type equilibrium structure, we have $u = 0$. For this market structure to be an equilibrium, we need $v_a > 0$, $v_a < 1$, the consumer $v = 1$ weakly preferring (B, AP) over (B, P), and the consumer $v = v_a$ weakly preferring (NB, NP) over (B, P).

The condition $v_a > 0$ is satisfied by our assumption that $\pi_a \alpha_a < 1$, since $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. Then for $v_a < 1$, we need $\delta p + c_a + \pi_a \alpha_a < 1$. For $v = 1$ to weakly prefer (B, AP) over (B, P), it needs to be the case that $v - \delta p - c_a - \pi_a \alpha_a v \geq v - p - c_p$ for $v = 1$. This simplifies to $c_p + (1 - \delta)p \geq c_a + \pi_a \alpha_a$. For $v = v_a$ to weakly prefer (NB, NP) over (B, P), it needs to be the case that $0 \geq v - p - c_p$ for $v = v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. This simplifies to $c_p \geq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$.

Next, for case (II), in which the top tier purchases (B, P) but the lower tier of consumers purchase (B, AP), i.e., $0 < v_a < v_p < 1$, we have $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}$. Following the same steps as before, we find the corresponding conditions for which case (II) arises. For this case to arise, we need $v_a > 0$, $v_p > v_a$, and $v_p < 1$. Again, $v_a > 0$ is satisfied under $\pi_a \alpha_a < 1$, one of the preliminary assumptions of the model. To have $v_p > v_a$, we need $\frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a} > \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. This simplifies to $c_p > \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$. Lastly, to have $v_p < 1$, we need $\frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a} < 1$. This simplifies to $c_p + (1 - \delta)p < c_a + \pi_a \alpha_a$.

Lastly, for case (III), in which consumers who purchase are all standard patching, choosing (B, P), $v_p = p + c_p$. For this case to be an equilibrium, we need $v_p > 0$, $v_p < 1$, $v = v_p$ preferring (NB, NP) over (B, AP), and $v = v_p$ preferring (B, P) over (B, AP). The condition $v_p > 0$ is satisfied. For $v_p < 1$, we need the condition $c_p + p < 1$. For $v = v_p$ to prefer (NB, NP) over (B, AP), we need $0 \geq v - \delta p - c_a - \pi_a \alpha_a v$ for $v = v_p = c_p + p$. This becomes $c_p + (1 - \delta)p \leq c_a + (c_p + p)\pi_a \alpha_a$. Lastly, for $v = v_p$ to prefer (B, P) over (B, AP), we need $v - p - c_p \geq v - \delta p - c_a - \pi_a \alpha_a v$ for $v = v_p = c_p + p$. This also simplifies to $c_p \leq \frac{c_a + p(\pi_a \alpha_a - (1 - \delta))}{1 - \pi_a \alpha_a}$. This concludes the proof of the consumer market equilibrium for the PPR case when $\pi_i \alpha_i > 1$. $\square$

Proof of Lemma 3: We prove that if $\pi_i \alpha_i > 1$ and patching rights are priced by the vendor, then if $c_p - \pi_a \alpha_a < c_a \leq c_p(1 - \pi_a \alpha_a)$, we have

$$p^* = \frac{1 - c_p}{2}, \tag{A.3}$$

$$\delta^* = \frac{1 - c_a - \pi_a \alpha_a}{1 - c_p}, \tag{A.4}$$

and σ^* is characterized by $0 < v_a < v_p < 1$ such that the lower tier of users prefers automated patching. On the other hand, if $c_a > c_p(1 - \pi_a \alpha_a)$, then

$$p^* = \frac{1 - c_p}{2}, \tag{A.5}$$

and σ^* is characterized by $0 < v_p < 1$ such that there is no user of automated patching in equilibrium.

Suppose $0 < v_a < 1$ is induced. Then the profit function is $\Pi_I(p, \delta) = \delta p(1 - v_a)$. Using Lemma A.2, we have that $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$. Similar to the status quo case, the optimal price and discount satisfies $\delta_1^* p_1^* = \frac{1 - c_a - \pi_a \alpha_a}{2}$ with the corresponding profit $\Pi_I^* = \frac{(1 - c_a - \pi_a \alpha_a)^2}{4(1 - \pi_a \alpha_a)}$. Notice in this case that there is not a unique maximizer, and in fact, the optimal (p^*, δ^*) traces out an isoprofit curve.

Next, suppose that $0 < v_a < v_p < 1$ is induced. Then the profit function is $\Pi_{II}(p, \delta) = p(1 - v_p) + \delta p(v_p - v_a)$. Using Lemma A.2, we have that $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}$. From the first-order condition for p, we have $p_2^*(\delta) = \frac{(1 - \pi_a \alpha_a)(-c_p(1 - \delta) + \pi_a \alpha_a) - c_a(1 - \delta - \pi_a \alpha_a)}{2((1 - \delta)^2 - \pi_a \alpha_a(1 - 2\delta))}$ with the second-order condition satisfied. Then maximizing $\Pi_{II}(p_2^*(\delta), \delta)$ with respect to δ, we find that $\delta_2^* = \frac{1 - c_a - \pi_a \alpha_a}{1 - c_p}$, and so $p_2^* = p_2^*(\delta_2^*) = \frac{1 - c_p}{2}$. The corresponding profit is $\Pi_{II} = \frac{1}{4}\left(1 - 2c_p + \frac{(c_a - c_p)^2}{\pi_a \alpha_a} + \frac{c_a^2}{1 - \pi_a \alpha_a}\right)$.

Lastly, suppose that $0 < v_p < 1$ is induced. Then the profit function is $\Pi_{III}(p, \delta) = p(1 - v_p)$. Using Lemma A.2, we have that $v_p = c_p + p$. As in the case when patching rights aren't priced, the optimal price is found to be $p_3^* = \frac{1 - c_p}{2}$ with the corresponding profit $\Pi_{III}^* = \frac{(1 - c_p)^2}{4}$.

We next find conditions under which the maximizing price for each case indeed induces that market structure. For $0 < v_a < 1$, we need the set of conditions for Case (I) in Lemma A.2 to hold for p_1^*. For $\delta p + c_a + \pi_a \alpha_a < 1$ to hold for the p_1^* and δ_1^*, we need $c_a + \pi_a \alpha_a < 1$, which is one of the preliminary assumptions of the model to not rule out automated patching for every consumer. Secondly, for $c_p + (1 - \delta)p \geq c_a + \pi_a \alpha_a$, we need $1 - c_a - \pi_a \alpha_a \geq \delta(1 + c_a + \pi_a \alpha_a - 2c_p)$. If $1 + c_a - 2c_p + \pi_a \alpha_a \leq 0$, then any δ satisfies this condition. Otherwise, we need $\delta \leq \frac{1 - c_a - \pi_a \alpha_a}{1 + c_a + \pi_a \alpha_a - 2c_p}$. Note that in this case, $\frac{1 - c_a - \pi_a \alpha_a}{1 + c_a + \pi_a \alpha_a - 2c_p} > 0$ so that such a δ (the corresponding $p_1^*(\delta)$) can be found. Last, we need $c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a$ to hold for the profit-maximizing p_1^* and δ_1^*. This

A.4

simplifies to $\delta(c_a+(1-2c_p)(1-\pi_a\alpha_a)) \le (1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)$. Then if $c_a+(1-2c_p)(1-\pi_a\alpha_a) \le 0$, any δ satisfies this condition. Otherwise, we'll need $\delta \le \frac{(1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)}{c_a+(1-2c_p)(1-\pi_a\alpha_a)}$. Note in this case that $\frac{(1-\pi_a\alpha_a)(1-c_a-\pi_a\alpha_a)}{c_a+(1-2c_p)(1-\pi_a\alpha_a)} > 0$ so that such a δ (and the corresponding $p_1^*(\delta)$) can be found. In summary, $0 < v_a < 1$ can always be induced in equilibrium by some p and δ, given a set of parameters $c_a, \pi_a\alpha_a$, and c_p that satisfy the preliminary model assumptions.

Similarly, for Case (II), the condition under which p_2^* would induce $0 < v_a < v_p < 1$ is $c_p - \pi_a\alpha_a < c_a \le c_p(1-\pi_a\alpha_a)$. And lastly, for Case (III), we need $\delta \ge \frac{-2c_a+(1+c_p)(1-\pi_a\alpha_a)}{1-c_p}$ for $c_p \le c_a + (1-\delta)p + (c_p+p)\pi_a\alpha_a$ to hold for $p = p_3^*$. This means that $0 < v_p < 1$ can always be induced in equilibrium using p_3^* by setting a high enough δ, given a set of parameters $c_a, \pi_a\alpha_a$, and c_p that satisfy the preliminary model assumptions.

Next, we find the conditions under which the maximal profits of each case dominate each other. First, note that $\Pi_I^* \ge \Pi_{III}^*$ iff $c_a \le 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$.

Next, note that $\Pi_{II}^* - \Pi_I^* = \frac{(c_a+\pi_a\alpha_a-c_p)^2}{4\pi_a\alpha_a}$, so that if $0 < v_a < v_p < 1$ can be induced, then it will dominate $0 < v_a < 1$. Also, $\Pi_{II}^* - \Pi_{III}^* = \frac{(c_a-c_p(1-\pi_a\alpha_a))^2}{4\pi_a\alpha_a(1-\pi_a\alpha_a)}$, so that if $0 < v_a < v_p < 1$ can be induced, then it will dominate $0 < v_p < 1$ as well. Therefore, when $c_p - \pi_a\alpha_a < c_a < c_p(1-\pi_a\alpha_a)$, then $0 < v_a < v_p < 1$ will be the equilibrium market structure.

Furthermore, consider the boundaries of this region. When $c_a = c_p - \pi_a\alpha_a$, then the profit of the adjacent region is $\Pi_I^* = \frac{(1-c_p)^2}{4(1-\pi_a\alpha_a)}$ while $\Pi_{II}^* = \frac{(1-c_p)^2}{4(1-\pi_a\alpha_a)}$ as well. Similarly, at the other end, when $c_a = c_p(1-\pi_a\alpha_a)$, then $\Pi_{III}^* = \frac{1}{4}(1-c_p)^2 = \Pi_{II}^*$. This means that $0 < v_a < 1$ will be the equilibrium market structure for $c_a \le c_p - \pi_a\alpha_a$ and $0 < v_p < 1$ will be the equilibrium market structure for $c_a \ge c_p(1-\pi_a\alpha_a)$. Note that if $c_a \ge c_p(1-\pi_a\alpha_a)$, then $\frac{-2c_a+(1+c_p)(1-\pi_a\alpha_a)}{1-c_p} \le 1$ so that $\delta_3^* = 1$ can be chosen to induce $0 < v_p < 1$ in equilibrium. $\square$

Proof of Proposition 1: We show that for $\pi_i\alpha_i > 1$, if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, the increase in profitability under PPR is given by

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1-\pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2}. \tag{A.6}$$

First, note that $1-\pi_a\alpha_a-(1-c_p)\sqrt{1-\pi_a\alpha_a} < c_p(1-\pi_a\alpha_a)$, since $0 < c_p < 1$ and $0 < \pi_a\alpha_a < 1$. Hence, when $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, in both the status quo case and when patching rights are priced, the equilibrium consumer market structure is $0 < v_a < v_p < 1$. Then from the proof of Lemma 3 above, the profit under PPR is $\Pi_P = \frac{1}{4}\left(1 - 2c_p + \frac{(c_a-c_p)^2}{\pi_a\alpha_a} + \frac{c_a^2}{1-\pi_a\alpha_a}\right)$ and from the proof of Lemma 2, the status quo case has $\Pi_{SQ} = \frac{(1-c_a-\pi_a\alpha_a)^2}{4(1-\pi_a\alpha_a)}$. Simplifying, we have $\frac{\Pi_P-\Pi_{SQ}}{\Pi_{SQ}} = \frac{(1-\pi_a\alpha_a)(c_a-c_p+\pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2}$. $\square$

B Low $\pi_i \alpha_i$

1 Status Quo

Lemma A.3 *Under the status quo, i.e., $\delta = 1$, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_a < 1)$, *where* $v_a = \frac{p + c_a}{1 - \pi_a \alpha_a}$:

    (A) $p + c_a + \pi_a \alpha_a < 1$

    (B) $c_p \geq c_a + \pi_a \alpha_a$

    (C) $(c_a + p)\pi_i \alpha_i \geq c_a + p(\pi_a \alpha_a)$

    (D) $c_p \geq c_a + (c_p + p)\pi_a \alpha_a$

    (E) $\pi_i \alpha_i > \pi_a \alpha_a$ *and* $c_a + p(\pi_a \alpha_a) \leq (c_a + p)\pi_i \alpha_i$

(II) $(0 < v_b < 1)$, *where* $v_b = \frac{1}{2} + \frac{-1 + \pi_i \alpha_i + \sqrt{(1 - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4p\pi_s \alpha_s}}{2\pi_s \alpha_s}$:

    (A) $1 - 2c_p + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$

    (B) $\pi_i \alpha_i < 1 - p$

    (C) $(1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) + (-1 + 2c_a + 2p + \pi_a \alpha_a)\pi_s \alpha_s \geq (1 - \pi_a \alpha_a)\sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$

    (D) *Either* $1 + \pi_i \alpha_i + \pi_s \alpha_s - 2\pi_a \alpha_a > 0$ *and* $p < \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}$ *and* $2(\pi_a \alpha_a + c_a) + \sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s} \geq \pi_i \alpha_i + \pi_s \alpha_s + 1$, *or*

        $\left(-2\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + 1 < 0 \text{ or } p > \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}\right)$ *and* $\pi_a \alpha_a (\pi_i \alpha_i + \pi_s \alpha_s) + 2\pi_s \alpha_s (c_a + p) + 1 \geq \pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + (1 - \pi_a \alpha_a)\sqrt{(\pi_i \alpha_i + \pi_s \alpha_s - 1)^2 + 4p\pi_s \alpha_s}$,

        or

        $p = \frac{(1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)}{\pi_s \alpha_s}$

(III) $(0 < v_b < v_a < 1)$, *where* v_b *is the most positive root of the cubic* $f_1(x) \triangleq (1 - \pi_a \alpha_a)\pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - p\pi_s \alpha_s)x^2 + (p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i))x + p^2$ *and* $v_a = \frac{c_a v_b}{v_b(1 - \pi_a \alpha_a) - p}$:

    (A) $(\pi_a \alpha_a + c_a - 1)(\pi_a \alpha_a - \pi_i \alpha_i + c_a) > \pi_s \alpha_s (\pi_a \alpha_a + c_a + p - 1)$

    (B) $\pi_i \alpha_i (c_a + p) < c_a + p(\pi_a \alpha_a)$

    (C) $\pi_a \alpha_a \leq c_p - c_a$

(IV) $(0 < v_b < v_a < v_p < 1)$, *where* v_b *is the most positive root of* $f_1(x)$ *and* $v_a = \frac{c_a v_b}{v_b(1 - \pi_a \alpha_a) - p}$ *and* $v_p = \frac{c_p - c_a}{\pi_a \alpha_a}$:

    (A) $c_p < c_a + \pi_a \alpha_a$

    (B) $c_p(1 - \pi_a \alpha_a) > c_a$

    (C) $c_p(\pi_a \alpha_a)^2(-c_a + c_p(1 - \pi_a \alpha_a)) + \pi_s \alpha_s(-c_a + c_p)^2(c_a - c_p(1 - \pi_a \alpha_a) + \pi_a \alpha_a p) - (c_a - c_p)(c_a + c_p(-1 + \pi_a \alpha_a))\pi_a \alpha_a \pi_i \alpha_i < 0$

    (D) $c_a + p(\pi_a \alpha_a) > (c_a + p)\pi_i \alpha_i$

(V) $(0 < v_b < v_p < 1)$, where v_b is the most positive root of $f_2(x) \triangleq \pi_s \alpha_s x^3 + (1 - \pi_i \alpha_i - (c_p + p) \pi_s \alpha_s) x^2 - p(2 - \pi_i \alpha_i) x + p^2$ and $v_p = \frac{c_p v_b}{v_b - p}$:

    (A) $(-1 + c_p + p) \pi_s \alpha_s < (1 - c_p)(-c_p + \pi_i \alpha_i)$

    (B) $\pi_i \alpha_i < \frac{c_p}{c_p + p}$

    (C) $(1 - \pi_a \alpha_a)(c_a + p\pi_a \alpha_a)(c_a + p\pi_a \alpha_a - (c_a + p)\pi_i \alpha_i) + (c_a + p)^2 (c_a - c_p + (c_p + p)\pi_a \alpha_a)\pi_s \alpha_s \geq 0$

    (D) $c_a + p(\pi_a \alpha_a) > 0$

    (E) Either $c_a + c_p(-1 + \pi_a \alpha_a) \geq 0$, or

$$c_a + c_p(-1 + \pi_a \alpha_a) < 0 \text{ and } \pi_a \alpha_a(c_a + c_p(-1 + \pi_a \alpha_a))(c_p \pi_a \alpha_a + (c_a - c_p)\pi_i \alpha_i) \leq$$
$$(-c_a + c_p)^2 (c_a - c_p + (c_p + p)\pi_a \alpha_a)\pi_s \alpha_s$$

(VI) $(0 < v_a < v_p < 1)$, where $v_a = \frac{p + c_a}{1 - \pi_a \alpha_a}$ and $v_p = \frac{c_p - c_a}{\pi_a \alpha_a}$:

    (A) $c_p < c_a + \pi_a \alpha_a$

    (B) $c_p > c_a + (c_p + p)\pi_a \alpha_a$

    (C) $(c_p - c_a)\pi_i \alpha_i \geq c_p \pi_a \alpha_a$

    (D) $\pi_i \alpha_i > \pi_a \alpha_a$ and $c_a + p\pi_a \alpha_a \leq (c_a + p)\pi_i \alpha_i$

(VII) $(0 < v_p < 1)$, where $v_p = p + c_p$:

    (A) $c_p + p < 1$

    (B) $c_a + (c_p + p)\pi_a \alpha_a \geq c_p$

    (C) $(c_p + p)\pi_i \alpha_i \geq c_p$

Proof of Lemma A.3: This is proven as a sub-case in the proof of Lemma A.4. $\square$

Proof of Lemma 4: Technically, we prove the existence of $\tilde{\alpha}_1$ such that if $\pi_i \alpha_i < \min\left[\pi_a \alpha_a, \frac{c_p}{1 + c_p}\right]$, then for $\alpha_s > \tilde{\alpha}_1$, p^* is set so that

1. if $c_p - \pi_a \alpha_a < c_a < 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and

2. if $c_a > 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

The sketch of the proof is as follows. From Lemma A.3, a unique consumer market equilibrium arises, given a price p. Within each region of the parameter space defined by Lemma A.3, the thresholds v_a, v_b, and v_p are smooth functions of the parameters, including p. In the cases where the thresholds are given in closed-form, this is clear. In the cases where these thresholds are implicitly defined as the root of some cubic equation, then the smoothness of the thresholds in the parameters follows from the Implicit Function Theorem. Specifically, for each of those cases, the threshold defined was the most positive root v_b^* of a cubic function of v_b, $f(v_b, p) = 0$. Moreover,

the cubic $f(v_b, p)$ has two local extrema in v_b and is negative to the left of v_b^* and positive to the right of it ($f(v_b^* - \epsilon, p) < 0$ and $f(v_b^* + \epsilon, p) > 0$ for arbitrarily small $\epsilon > 0$). Therefore, $\frac{\partial f}{\partial v_b}(v_b, p) \neq 0$ so that the Implicit Function Theorem applies. The thresholds being smooth in p implies that the profit function for each case of the parameter space defined by Lemma A.3 is smooth in p. We find the profit-maximizing price within the compact closure of each case, so that the price that induces the largest profit among the cases will be the equilibrium price set by the vendor.

Having given the sketch of the proof, we now proceed with the proof. The conditions of this lemma precludes candidate market structures from arising in equilibrium. Specifically, $c_p - \pi_a \alpha_a < c_a$ rules out Cases (I) and (III) of Lemma A.3, and $\pi_i \alpha_i < \min\left[\frac{c_p \pi_a \alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$ rules out Cases (VI) and (VII). We consider the remaining possible consumer equilibria that can be induced when the vendor sets prices optimally. Suppose $0 < v_b < 1$ is induced. By part (II) of Lemma A.3, we obtain $v_b = \frac{1}{2} + \frac{-1 + \pi_i \alpha_i + \sqrt{(1 - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4p\pi_s \alpha_s}}{2\pi_s \alpha_s}$. The profit function in this case is $\Pi(p) = p(1 - v_b(p))$. Let C_{II} be the compact closure of the region of the parameter space defining $0 < v_b < 1$, given in part (II) of Lemma A.3. By the Weierstrass extreme value theorem, there exists a p in C_{II} that maximizes $\Pi(p)$. This p may be on the boundary, and we show that the vendor's profit function is continuous across region boundaries later. Otherwise, if this p is interior, the unconstrained maximizer satisfies the first-order condition $\Pi'(p) = 0$.

Using the first-order condition and letting

$$Q_1 \triangleq \sqrt{(-\pi_i \alpha_i + \pi_s \alpha_s + 1)^2 \left(\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2\right)},$$

the roots of $\Pi'(p) = 0$ are $\frac{-(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_s \alpha_s) + \pi_s \alpha_s (4 - \pi_s \alpha_s) - 1}{9\pi_s \alpha_s} \mp \frac{Q_1}{9\pi_s \alpha_s}$.

However, $\frac{-(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_s \alpha_s) + \pi_s \alpha_s (4 - \pi_s \alpha_s) - 1 - Q_1}{9\pi_s \alpha_s} < 0$ for $\pi_s \alpha_s > 1 - \pi_i \alpha_i$, so for $\pi_s \alpha_s > 1 - \pi_i \alpha_i$, the unconstrained maximizer is given by

$$p_{II} = \frac{-(\pi_i \alpha_i)^2 + 2\pi_i \alpha_i (1 - 2\pi_s \alpha_s) + \pi_s \alpha_s (4 - \pi_s \alpha_s) - 1 + Q_1}{9\pi_s \alpha_s}. \tag{A.7}$$

The second-order condition is satisfied if $Q_1 + 2\left(\pi_s \alpha_s (\pi_i \alpha_i - 1) + (\pi_i \alpha_i - 1)^2 + (\pi_s \alpha_s)^2\right) > 0$, which holds when $\pi_s \alpha_s > 1 - \pi_i \alpha_i$. Substituting (A.7) into the profit function, we obtain

$$\Pi_{II} = \frac{1}{54(\pi_s \alpha_s)^2}\left(\left((\pi_i \alpha_i)^2 - Q_1 + 2\pi_i \alpha_i (2\pi_s \alpha_s - 1) + \pi_s \alpha_s (\pi_s \alpha_s - 4) + 1\right)\right.$$
$$\left.\left(\sqrt{5(\pi_i \alpha_i)^2 + 4Q_1 + 2\pi_i \alpha_i (\pi_s \alpha_s - 5) + \pi_s \alpha_s (5\pi_s \alpha_s - 2) + 5} + 3\pi_i \alpha_i - 3\pi_s \alpha_s - 3\right)\right). \tag{A.8}$$

On the other hand, suppose $0 < v_b < v_a < v_p < 1$ is induced. By part (IV) of Lemma A.3, we obtain that v_b is the most positive root of the cubic

$$f_1(x) \triangleq (1 - \pi_a \alpha_a)\pi_s \alpha_s x^3 + ((1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) - c_a \pi_s \alpha_s - p\pi_s \alpha_s)x^2 +$$
$$(p(-1 + \pi_a \alpha_a) + p(-1 + \pi_i \alpha_i))x + p^2. \tag{A.9}$$

A.8

The profit function is $\Pi_{IV}(p) = p(1 - v_b(p))$. Let C_{IV} be the compact closure of the region of the parameter space defining $0 < v_b < v_a < v_p < 1$, given in part (IV) of Lemma A.3. Again, by the Weierstrass extreme value theorem, there exists a p in C_{IV} that maximizes $\Pi(p)$. This p may be on the boundary, and we show that the vendor's profit function is continuous across region boundaries later. Otherwise, if this p is interior, the unconstrained maximizer satisfies the first-order condition $\Pi'(p) = 0$. The first-order condition is given by

$$\Pi'_{IV}(p) = (1 - v_b(p)) - pv'_b(p) = 0. \tag{A.10}$$

By equating (A.9) to 0 and implicitly differentiating, we have that

$$v'_b(p) = \frac{v_b(p)(\pi_a\alpha_a + \pi_i\alpha_i - \pi_s\alpha_s v_b(p) - 2) + 2p}{v_b(p)(-2(\pi_a\alpha_a - 1)(\pi_i\alpha_i - 1) + 2\pi_s\alpha_s(c_a + p) + 3\pi_s\alpha_s(\pi_a\alpha_a - 1)v_b(p)) - p(\pi_a\alpha_a + \pi_i\alpha_i - 2)} \tag{A.11}$$

Substituting this into (A.10) and re-writing (A.9), we have that $v_b(p^*)$ and p^* simultaneously need to solve

$$(1 - \pi_a\alpha_a)\pi_s\alpha_s v_b^3 + ((1 - \pi_a\alpha_a)(1 - \pi_i\alpha_i) - c_a\pi_s\alpha_s - p\pi_s\alpha_s)v_b^2 +$$
$$(p(-1 + \pi_a\alpha_a) + p(-1 + \pi_i\alpha_i))v_b + p^2 = 0, \text{ and } \tag{A.12}$$

$$1 - v_b - \frac{p(2p + v_b(\pi_a\alpha_a + \pi_i\alpha_i - \pi_s\alpha_s v_b - 2))}{v_b(-2(\pi_a\alpha_a - 1)(\pi_i\alpha_i - 1) + 2\pi_s\alpha_s(c_a + p) + 3\pi_s\alpha_s v_b(\pi_a\alpha_a - 1)) - p(\pi_a\alpha_a + \pi_i\alpha_i - 2)} = 0. \tag{A.13}$$

Letting

$$Q_2 \triangleq \sqrt{(\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s(v_b - 2)v_b - 2)^2 - 8(v_b - 1)v_b(\pi_s\alpha_s(2c_a + 3v_b(\pi_a\alpha_a - 1)) - 2(\pi_a\alpha_a - 1)(\pi_i\alpha_i - 1))}$$

and solving (A.13) for p, we have that p is either $\frac{1}{4}\left(-\pi_a\alpha_a - \pi_i\alpha_i - \pi_s\alpha_s v_b^2 + 2\pi_s\alpha_s v_b + 2 - Q_2\right)$ or $\frac{1}{4}\left(-\pi_a\alpha_a - \pi_i\alpha_i - \pi_s\alpha_s v_b^2 + 2\pi_s\alpha_s v_b + 2 + Q_2\right)$. We can rule out the larger root when $\pi_s\alpha_s > 2 + \pi_a\alpha_a + \pi_i\alpha_i$ since when $\pi_s\alpha_s > 2 + \pi_a\alpha_a + \pi_i\alpha_i$, then $Q_2 > -2 + 4v_b + \pi_a\alpha_a + \pi_i\alpha_i - (2 - v_b)v_b\pi_s\alpha_s$. This is equivalent to the larger root for p^* being greater than v_b, which can't happen in equilibrium. Therefore,

$$p(v_b) = \frac{1}{4}\left(-\pi_a\alpha_a - \pi_i\alpha_i - \pi_s\alpha_s v_b^2 + 2\pi_s\alpha_s v_b + 2 - Q_2\right) \tag{A.14}$$

Substituting this into (A.12), we have that $v_b(p^*)$ solves

$$\left((\pi_a\alpha_a + \pi_i\alpha_i - 2)^2 + 4v_b^2(\pi_a\alpha_a(4\pi_i\alpha_i + 5\pi_s\alpha_s - 4) + 2\pi_i\alpha_i(\pi_s\alpha_s - 2) + \pi_s\alpha_s(\pi_s\alpha_s - 4c_a - 7) + 4) - \right.$$
$$4\pi_s\alpha_s v_b(\pi_a\alpha_a + \pi_i\alpha_i - 2c_a - 2) + 3(\pi_s\alpha_s)^2 v_b^4 - 2\pi_s\alpha_s v_b^3(11\pi_a\alpha_a + \pi_i\alpha_i + 4\pi_s\alpha_s - 12) - 16v_b -$$
$$\left.2v_b\left((\pi_a\alpha_a)^2 + 2\pi_a\alpha_a(3\pi_i\alpha_i - 4) + \pi_i\alpha_i(\pi_i\alpha_i - 8)\right)\right) + (\pi_s\alpha_s v_b(3v_b - 2) - (2v_b - 1)(\pi_a\alpha_a + \pi_i\alpha_i - 2))Q_2 = 0 \tag{A.15}$$

A generalization of the Implicit Function Theorem gives that v_b is not only a smooth function of the parameters, but it's also an analytic function of the parameters so that it can be represented

locally as a Taylor series of its parameters (Brillinger 1966). More specifically, since $f_1'(x) \neq 0$ at the root for which v_b is defined, there exists an $\alpha_1 > 0$ such that for $\alpha_s > \alpha_1$, $v_b = \sum_{k=0}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k}$ for some coefficients a_k .

Substituting this into (A.15), we have $\pi_s \alpha_s \left(\frac{a_0^3((2a_0 - 1)\pi_a \alpha_a - 2a_0 + c_a + 1)}{a_0 - 2} \right) + \sum_{k=0}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0$. Then $a_0 = 0$ or $a_0 = \frac{1 + c_a - \pi_a \alpha_a}{2(1 - \pi_a \alpha_a)}$ are the only solutions for a_0 that make the first term 0. However, if $a_0 = 0$, then $\pi_s \alpha_s \left(\frac{a_0^3((2a_0 - 1)\pi_a \alpha_a - 2a_0 + c_a + 1)}{a_0 - 2} \right) + \sum_{k=0}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k}$ becomes $c_a^2 + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k}$. Note that $c_a^2 + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} > 0$ for large enough $\pi_s \alpha_s$ so that there exists no coefficients A_k such that $c_a^2 + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0$ for $\alpha_s > \alpha_1$, a contradiction.

Thus $a_0 = \frac{1 + c_a - \pi_a \alpha_a}{2(1 - \pi_a \alpha_a)}$. Substituting $v_b = \frac{1 + c_a - \pi_a \alpha_a}{2(1 - \pi_a \alpha_a)} + \sum_{k=1}^{\infty} a_k \xi^k$ into (A.15), we have that $\frac{a_1(-\pi_a \alpha_a + c_a + 1)^3 + 2c_a(\pi_a \alpha_a - 1)\left((\pi_a \alpha_a - 1)(\pi_a \alpha_a - \pi_i \alpha_i) + c_a^2 + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3)\right)}{2(\pi_a \alpha_a - 1)(3\pi_a \alpha_a + c_a - 3)} + \sum_{k=1}^{\infty} \frac{A_k}{(\pi_s \alpha_s)^k} = 0$. Solving for a_1 to make this first term zero, we have $a_1 = \frac{2c_a(1 - \pi_a \alpha_a)\left((1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3)\right)}{(-\pi_a \alpha_a + c_a + 1)^3}$.

Using $v_b = \frac{1 + c_a - \pi_a \alpha_a}{2(1 - \pi_a \alpha_a)} + \frac{2c_a(1 - \pi_a \alpha_a)\left((1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3)\right)}{(-\pi_a \alpha_a + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} \frac{a_k}{(\pi_s \alpha_s)^k}$, we can solve for a_2, a_3, and so on recursively by repeatedly substituting this expression for v_b into (A.15) and solving for the coefficients to make the expression zero. Doing this, we find that the threshold v_b is

$$ v_b = \frac{1 + c_a - \pi_a \alpha_a}{2(1 - \pi_a \alpha_a)} + \frac{2c_a(1 - \pi_a \alpha_a)\left((1 - \pi_a \alpha_a)(-\pi_a \alpha_a + \pi_i \alpha_i) + c_a^2 + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3)\right)}{(-\pi_a \alpha_a + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s \alpha_s} \right)^k, $$

(A.16)

and substituting this into (A.14), we have that the optimal price set by the vendor is

$$ p_{IV}^* = \frac{1}{2}(1 - \pi_a \alpha_a - c_a) + \frac{2c_a^2(\pi_a \alpha_a - 1)((\pi_a \alpha_a - 1)(2\pi_a \alpha_a - \pi_i \alpha_i - 1) + c_a(2\pi_a \alpha_a + \pi_i \alpha_i - 3))}{(-\pi_a \alpha_a + c_a + 1)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} b_k \left(\frac{1}{\pi_s \alpha_s} \right)^k. $$

(A.17)

The corresponding profit is given as

$$ \Pi_{IV}^* = \frac{(\pi_a \alpha_a + c_a - 1)^2}{4(1 - \pi_a \alpha_a)} + \frac{c_a(\pi_a \alpha_a + c_a - 1)((\pi_a \alpha_a - 1)(\pi_a \alpha_a - \pi_i \alpha_i) + c_a(\pi_a \alpha_a + \pi_i \alpha_i - 2))}{(-\pi_a \alpha_a + c_a + 1)^2 \pi_s \alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s \alpha_s} \right)^k. $$

(A.18)

As a matter of notation, we will use $a_k, b_k,$ and c_k to denote coefficients in the Taylor expansions without referring to specific expressions throughout the appendix. These will be used across different cases, and they don't refer to the same quantities or expressions across cases.

Lastly, suppose $0 < v_b < v_p < 1$ is induced. The profit function in this case is $\Pi_V(p) = p(1 - v_b(p))$, where v_b is the most positive root of $f_2(x) \triangleq \pi_s \alpha_s x^3 + (1 - \pi_i \alpha_i - (c_p + p)\pi_s \alpha_s)x^2 - p(2 - \pi_i \alpha_i)x + p^2$ by part (V) of Lemma A.3. Omitting the algebra (similar to the previous case), there exists an $\alpha_2 > 0$ such that for $\alpha_s > \alpha_2$, the unconstrained maximizer is given as

$$ p_V^* = \frac{1 - c_p}{2} - \frac{2c_p^2(1 - 3c_p + \pi_i \alpha_i(1 + c_p))}{(1 + c_p)^3 \pi_s \alpha_s} + \sum_{k=2}^{\infty} b_k \left(\frac{1}{\pi_s \alpha_s} \right)^k, \qquad \text{(A.19)} $$

A.10

and the profit induced is given as

$$\Pi_V = \frac{1}{4}(1 - c_p)^2 + \frac{(1 - c_p)c_p\left(2c_p - \pi_i\alpha_i(1 + c_p)\right)}{(1 + c_p)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.20}$$

To compare profits, we note that there exists $\alpha_3 > 0$ such that for $\alpha_s > \alpha_3$, (A.8) can be expressed as the Taylor series

$$\Pi_{II} = \frac{(1 - \pi_i\alpha_i)^2}{4\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.21}$$

Then comparing (A.21) with either (A.18) or (A.20), we find that (A.8) is dominated by the other two profits when α_s exceeds an implicit bound (say, $\alpha_s > \hat{\alpha}_1$, for some $\hat{\alpha}_1 > 0$).

Next, using (A.17) with Lemma A.3, we find the conditions under which the interior optimal price of $0 < v_b < v_a < v_p < 1$ would indeed induce this market structure. First, we still need the conditions $c_p < c_a + \pi_a\alpha_a$ and $c_p(1 - \pi_a\alpha_a) > c_a$. Next, for $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1 - \pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + c_p)^2(c_a - c_p(1 - \pi_a\alpha_a) + \pi_a\alpha_a p) - (c_a - c_p)(c_a + c_p(-1 + \pi_a\alpha_a))\pi_a\alpha_a\pi_i\alpha_i < 0$ to hold for $p = p_{IV}^*$, we need $-\frac{1}{2}\left((c_p - c_a)^2(-2c_a + 2c_p + (-1 + c_a - 2c_p)\pi_a\alpha_a + (\pi_a\alpha_a)^2)\right)\pi_s\alpha_s + \sum_{k=0}^{\infty} A_k \left(\frac{1}{\pi_s\alpha_s}\right)^k < 0$

for some coefficients A_k. There exists $\alpha_4 > 0$ such that if $\alpha_s > \alpha_4$, then $c_a < \frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a}$ is sufficient for this condition to hold. Note that $\frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a} < c_p(1 - \pi_a\alpha_a)$, so $c_a < \frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a}$ is the tighter bound on c_a. Lastly, for $c_a + p(\pi_a\alpha_a) > (c_a + p)\pi_i\alpha_i$ to hold for $p = p_{IV}^*$, we need $c_a - \frac{1}{2}\pi_a\alpha_a(-1 + c_a + \pi_a\alpha_a) - \frac{1}{2}\pi_i\alpha_i(1 + c_a - \pi_a\alpha_a) + \sum_{k=0}^{\infty} B_k \left(\frac{1}{\pi_s\alpha_s}\right)^k > 0$ for some

coefficients $B_k > 0$. It suffices to have $c_a > \frac{(1 - \pi_a\alpha_a)(\pi_a\alpha_a - \pi_i\alpha_i)}{(-2 + \pi_a\alpha_a + \pi_i\alpha_i)}$. Since $\pi_i\alpha_i < \pi_a\alpha_a$ follows from one of the assumptions of this lemma ($\pi_i\alpha_i < \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}$), this condition is automatically satisfied since $c_a > 0$.

In summary, the optimal price $0 < v_b < v_a < v_p < 1$ indeed induces this market structure when $c_p - \pi_a\alpha_a < c_a < \frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a}$ for $\alpha > \alpha_4$.

Similarly, using (A.19) with Lemma A.3, there exists $\alpha_5 > 0$ such that when $\alpha_s > \alpha_5$, the optimal price of $0 < v_b < v_p < 1$ indeed induces the correct market structure when $c_a > c_p - \frac{1}{2}(1 + c_p)\pi_a\alpha_a$ and $\pi_i\alpha_i < \frac{2c_p}{1 + c_p}$.

Let $\tilde{\alpha}_1$ be the max of $\hat{\alpha}_1$, α_1, α_2, α_3, α_4, and α_5. Then for $\alpha_s > \tilde{\alpha}_1$, if $c_a > c_p - \pi_a\alpha_a$, we have that (A.8) is dominated. Moreover, since $c_p - \frac{1}{2}(1 + c_p)\pi_a\alpha_a < \frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a}$, there will be a region in which the interior maximizers of both $0 < v_b < v_a < v_p < 1$ and $0 < v_b < v_p < 1$ induce their corresponding cases.

Comparing (A.18) with (A.20), we see that (A.18) is greater when $2c_a(1 - \pi_a\alpha_a) < c_a^2 + (1 - \pi_a\alpha_a)(c_p(2 - c_p) - \pi_a\alpha_a)$, which can be written as $c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$. Note that for any $c_p \in [0, 1]$, we have that $c_p - \frac{1}{2}(1 + c_p)\pi_a\alpha_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a} < \frac{(2c_p - \pi_a\alpha_a)(1 - \pi_a\alpha_a)}{2 - \pi_a\alpha_a}$ from $\pi_a\alpha_a \in (0, 1)$. Therefore, for $\alpha_s > \tilde{\alpha}_1$, if $c_p - \pi_a\alpha_a < c_a \le 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_a < 1$ under optimal pricing, and if $c_a > 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing. $\square$

2 Pricing Patching Rights

Lemma A.4 *Under PPR, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_a < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$:

    (A) $\delta p + c_a + \pi_a \alpha_a < 1$

    (B) $(1 - \delta)p + c_p \geq c_a + \pi_a \alpha_a$

    (C) $(c_a + \delta p)\pi_i \alpha_i \geq c_a + p(-1 + \delta + \pi_a \alpha_a)$

    (D) $c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a$

    (E) *Either* $\pi_i \alpha_i < \pi_a \alpha_a$ *and* $\pi_a \alpha_a - \pi_i \alpha_i + c_a \leq (1 - \delta)p$, *or*

            $\pi_i \alpha_i > \pi_a \alpha_a$ *and* $c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i$, *or*

            $\pi_i \alpha_i = \pi_a \alpha_a$ *and* $(1 - \delta)p - c_a \geq 0$

(II) $(0 < v_a < v_b < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$ *and* $v_b = \frac{-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s (c_a + (\delta - 1)p)}}{2\pi_s \alpha_s}$:

    (A) $(1 - \delta)p - c_a > 0$

    (B) $\frac{(2\pi_s \alpha_s)(c_a + \delta p)}{1 - \pi_a \alpha_a} < -\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s (c_a + (\delta - 1)p)}$

    (C) $(1 - \delta)p < \pi_a \alpha_a - \pi_i \alpha_i + c_a$

    (D) $\pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s > 0$

    (E) $\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s (c_a + (\delta - 1)p)} + 2c_p$

(III) $(0 < v_a < v_b < v_p < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}$, v_b *is the most positive root of the cubic* $f_3(x) \triangleq \pi_s \alpha_s x^2 (c_a - c_p + (\delta - 1)p + \pi_a \alpha_a x) + (c_a + (\delta - 1)p + \pi_a \alpha_a x)(c_a + (\delta - 1)p + x(\pi_a \alpha_a - \pi_i \alpha_i))$, *and* $v_p = \frac{c_p v_b}{c_a - (1 - \delta)p + \pi_a \alpha_a v_b}$:

    (A) $(1 - \delta)p - c_a \geq 0$

    (B) $\pi_a \alpha_a > c_p$

    (C) $\pi_s \alpha_s (-\pi_a \alpha_a - c_a + c_p - \delta p + p) < (c_p - \pi_a \alpha_a)(c_p - \pi_i \alpha_i)$

    (D) $\pi_i \alpha_i (c_a + \delta p) + \pi_a \alpha_a c_p > \pi_i \alpha_i (c_p + p)$

    (E) *Either* $c_a + p(\pi_a \alpha_a + \delta) \leq p$, *or*

           $c_a + p(\pi_a \alpha_a + \delta) > p$ *and* $\pi_s \alpha_s (c_a + \delta p)^2 (c_a + \pi_a \alpha_a (c_p + p) - c_p + (\delta - 1)p) + (\pi_a \alpha_a - 1)(c_a + p(\pi_a \alpha_a + \delta - 1))(\pi_i \alpha_i (c_a + \delta p) - c_a - p(\pi_a \alpha_a + \delta - 1)) > 0$

(IV) $(0 < v_b < 1)$, *where* $v_b = \frac{1}{2} + \frac{-1 + \pi_i \alpha_i + \sqrt{(1 - \pi_s \alpha_s - \pi_i \alpha_i)^2 + 4p\pi_s \alpha_s}}{2\pi_s \alpha_s}$:

    (A) $1 - 2c_p + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$

    (B) $\pi_i \alpha_i < 1 - p$

    (C) $(1 - \pi_a \alpha_a)(1 - \pi_i \alpha_i) + (-1 + 2c_a + 2\delta p + \pi_a \alpha_a)\pi_s \alpha_s \geq (1 - \pi_a \alpha_a)\sqrt{4p\pi_s \alpha_s + (1 - \pi_s \alpha_s - \pi_i \alpha_i)^2}$

(D) *Either* $1 + \pi_i\alpha_i + \pi_s\alpha_s - 2\pi_a\alpha_a > 0$ *and* $p < \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s}$ *and* $2(\pi_a\alpha_a +$
$c_a - (1-\delta)p) + \sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s} \geq \pi_i\alpha_i + \pi_s\alpha_s + 1$, *or*

$$\left(-2\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + 1 < 0 \text{ or } p > \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s}\right) \text{ and } \pi_a\alpha_a(\pi_i\alpha_i +$$
$\pi_s\alpha_s) + 2\pi_s\alpha_s(c_a + \delta p) + 1 \geq \pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + (1-\pi_a\alpha_a)\sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s}$,
or
$$p = \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s} \text{ and } (1-\delta)p - c_a \leq 0$$

(V) $(0 < v_b < v_a < 1)$, *where* v_b *is the most positive root of the cubic* $f_4(x) \triangleq (1-\pi_a\alpha_a)\pi_s\alpha_s x^3 +$
$((1-\pi_a\alpha_a)(1-\pi_i\alpha_i) - c_a\pi_s\alpha_s - \delta p\pi_s\alpha_s)x^2 + (p(-1+\pi_a\alpha_a) + p(-1+\pi_i\alpha_i))x + p^2$ *and*
$v_a = \frac{(c_a-(1-\delta)p)v_b}{v_b(1-\pi_a\alpha_a)-p}$:

(A) $(1-\delta)p - c_a < 0$

(B) $(\pi_a\alpha_a + c_a - 1 - (1-\delta)p)(\pi_a\alpha_a - \pi_i\alpha_i + c_a - (1-\delta)p) > \pi_s\alpha_s(\pi_a\alpha_a + c_a + \delta p - 1)$

(C) $\pi_i\alpha_i(c_a + \delta p) < c_a + p(-1 + \delta + \pi_a\alpha_a)$

(D) $\pi_a\alpha_a \leq (1-\delta)p + c_p - c_a$

(VI) $(0 < v_b < v_a < v_p < 1)$, *where* v_b *is the most positive root of* $f_4(x)$ *and* $v_a = \frac{(c_a-(1-\delta)p)v_b}{v_b(1-\pi_a\alpha_a)-p}$
and $v_p = \frac{(1-\delta)p+c_p-c_a}{\pi_a\alpha_a}$:

(A) $(1-\delta)p - c_a < 0$

(B) $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$

(C) $c_p(1-\pi_a\alpha_a) + (1-\delta)p > c_a$

(D) $c_p(\pi_a\alpha_a)^2(-c_a + (1-\delta)p + c_p(1-\pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + (1-\delta)p + c_p)^2(c_a - c_p(1-\pi_a\alpha_a) +$
$(\pi_a\alpha_a - 1 + \delta)p) - (c_a - c_p - (1-\delta)p)(c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a))\pi_a\alpha_a\pi_i\alpha_i < 0$

(E) $c_a + p(-1 + \delta + \pi_a\alpha_a) > (c_a + \delta p)\pi_i\alpha_i$

(VII) $(0 < v_b < v_p < 1)$, *where* v_b *is the most positive root of* $f_5(x) \triangleq \pi_s\alpha_s x^3 + (1 - \pi_i\alpha_i - (c_p +$
$p)\pi_s\alpha_s)x^2 - p(2 - \pi_i\alpha_i)x + p^2$ *and* $v_p = \frac{c_p v_b}{v_b - p}$:

(A) $(-1 + c_p + p)\pi_s\alpha_s < (1 - c_p)(-c_p + \pi_i\alpha_i)$

(B) $\pi_i\alpha_i < \frac{c_p}{c_p + p}$

(C) $(1 - \pi_a\alpha_a)(c_a + (\pi_a\alpha_a - (1-\delta))p)(c_a + p(\pi_a\alpha_a - (1-\delta)) - (c_a + \delta p)\pi_i\alpha_i) + (c_a + \delta p)^2(c_a -$
$c_p - (1-\delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s \geq 0$

(D) $c_a + p(\delta + \pi_a\alpha_a) > p$

(E) *Either* $c_a - (1-\delta)p + c_p(-1 + \pi_a\alpha_a) \geq 0$, *or*
$c_a + c_p(-1 + \pi_a\alpha_a) < 0$ *and* $\pi_a\alpha_a(c_a - (1-\delta)p + c_p(-1 + \pi_a\alpha_a))(c_p\pi_a\alpha_a + (c_a -$
$c_p - (1-\delta)p)\pi_i\alpha_i) \leq (-c_a + c_p + (1-\delta)p)^2(c_a - c_p - (1-\delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s$

(VIII) $(0 < v_a < v_p < 1)$, *where* $v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}$ *and* $v_p = \frac{(1-\delta)p+c_p-c_a}{\pi_a\alpha_a}$:

(A) $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$

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(B) $c_p + (1 - \delta)p > c_a + (c_p + p)\pi_a \alpha_a$

(C) $(c_p - c_a + (1 - \delta)p)\pi_i \alpha_i \geq c_p \pi_a \alpha_a$

(D) Either $\pi_i \alpha_i < \pi_a \alpha_a$ and $c_p \pi_a \alpha_a + \pi_i \alpha_i (c_a - c_p - (1 - \delta)p) \leq 0$, or

$\quad\quad \pi_i \alpha_i > \pi_a \alpha_a$ and $c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i$, or

$\quad\quad \pi_i \alpha_i = \pi_i \alpha_i$ and $(1 - \delta)p - c_a \geq 0$

(IX) $(0 < v_p < 1)$, where $v_p = p + c_p$:

(A) $c_p + p < 1$

(B) $c_a + (c_p + p)\pi_a \alpha_a \geq c_p + (1 - \delta)p$

(C) $(c_p + p)\pi_i \alpha_i \geq c_p$

Proof of Lemma A.4: First, we establish the general threshold-type equilibrium structure. Given the size of unpatched user population u, the net payoff of the consumer with type v for strategy profile σ is written as

$$U(v, \sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (B, P); \\ v - p - \pi_s \alpha_s uv - \pi_i \alpha_i v & if \quad \sigma(v) = (B, NP); \\ v - \delta p - c_a - \pi_a \alpha_a v & if \quad \sigma(v) = (B, AP); \\ 0 & if \quad \sigma(v) = (NB, NP). \end{cases} \quad (A.22)$$

Note $\sigma(v) = (B, P)$ if and only if

$$v - p - c_p \geq v - p - \pi_s \alpha_s uv - \pi_i \alpha_i v \Leftrightarrow v \geq \frac{c_p}{\pi_s \alpha_s u + \pi_i \alpha_i}, \text{ and}$$

$$v - p - c_p \geq v - \delta p - c_a - \pi_a \alpha_a v \Leftrightarrow v \geq \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}, \text{ and}$$

$$v - p - c_p \geq 0 \Leftrightarrow v \geq c_p + p,$$

which can be summarized as

$$v \geq \max \left(\frac{c_p}{\pi_s \alpha_s u + \pi_i \alpha_i}, \frac{(1 - \delta)p + c_p - c_a}{\pi_a \alpha_a}, c_p + p \right). \quad (A.23)$$

By (A.23), if a consumer with valuation v_0 buys and patches the software, then every consumer with valuation $v > v_0$ will also do so. Hence, there exists a threshold $v_p \in (0, 1]$ such that for all $v \in \mathcal{V}$, $\sigma^*(v) = (B, P)$ if and only if $v \geq v_p$. Similarly, $\sigma(v) \in \{(B, P), (B, NP), (B, AP)\}$, i.e., the consumer purchases one of the alternatives, if and only if

$$v - p - c_p \geq 0 \Leftrightarrow v \geq c_p + p, \text{ or}$$

$$v - p - \pi_s \alpha_s uv - \pi_i \alpha_i v \geq 0 \Leftrightarrow v \geq \frac{p}{1 - \pi_s \alpha_s u - \pi_i \alpha_i}, \text{ or}$$

$$v - \delta p - c_a - \pi_a \alpha_a v \geq 0 \Leftrightarrow v \geq \frac{\delta p + c_a}{1 - \pi_a \alpha_a},$$

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which can be summarized as

$$v \geq \min\left(c_p + p, \frac{p}{1 - \pi_s \alpha_s u - \pi_i \alpha_i}, \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\right). \tag{A.24}$$

Let $0 < v_1 \leq 1$ and $\sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\}$, then by (A.24), for all $v > v_1$, $\sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\}$, and hence there exists a $\underline{v} \in (0, 1]$ such that a consumer with valuation $v \in \mathcal{V}$ will purchase if and only if $v \geq \underline{v}$.

By (A.23) and (A.24), $\underline{v} \leq v_p$ holds. Moreover, if $\underline{v} < v_p$, consumers with types in $[\underline{v}, v_p]$ choose either (B, NP) or (B, AP). A purchasing consumer with valuation v will prefer (B, NP) over (B, AP) if and only if

$$v - p - \pi_s \alpha_s u v - \pi_i \alpha_i v > v - \delta p - c_a - \pi_a \alpha_a v \Leftrightarrow v(\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i) > (1 - \delta)p - c_a. \tag{A.25}$$

This inequality can be either $v > \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i}$ or $v < \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i}$, depending on the sign of $\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i$. Consequently, there can be two cases for (B, NP) and (B, AP) in equilibrium: first, there exists $v_u \in [\underline{v}, v_p]$ such that $\sigma(v) = (B, NP)$ for all $v \in [v_u, v_p)$, and $\sigma(v) = (B, AP)$ for all $v \in [v_d, v_u)$ where $v_d = \underline{v}$. In the second case, there exists $v_d \in [\underline{v}, v_p]$ such that $\sigma(v) = (B, AP)$ for all $v \in [v_d, v_p)$, and $\sigma(v) = (B, NP)$ for all $v \in [v_u, v_d)$, where $v_u = \underline{v}$. If $\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i = 0$, then depending on the sign of $(1-\delta)p - c_a$, all consumers unilaterally prefer either (B, NP) or (B, AP); e.g., if $(1-\delta)p > c_a$, all consumers prefer (B, AP), and if $(1-\delta)p < c_a$, then all consumers prefer (B, NP). Finally, if $(1 - \delta)p = c_a$, then all consumers are indifferent between (B, NP) and (B, AP), in which case only the size of the consumer population u matters in equilibrium, i.e., $\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i = 0$ in equilibrium. Technically, there are multiple equilibria in this case; however, utility of each consumer and the vendor's profit are the same in all equilibria. So, without loss of generality, we focus on the threshold-type equilibrium in this case. In summary, we have established the threshold-type consumer market equilibrium structure.

Next, we characterize in more detail each outcome that can arise in equilibrium, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose the automated patching option, i.e., $0 < v_a < 1$, based on the threshold-type equilibrium structure, we have $u = 0$. We prove the following claim related to the corresponding parameter region in which Case (I) arises.

Claim 3 *The equilibrium that corresponds to case (I) arises if and only if the following conditions are satisfied:*

$$\delta p + c_a + \pi_a \alpha_a < 1 \ \text{and} \ (1 - \delta)p + c_p \geq c_a + \pi_a \alpha_a \ \text{and}$$

$$(c_a + \delta p)\pi_i \alpha_i \geq c_a + p(-1 + \delta + \pi_a \alpha_a) \ \text{and} \ c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a \ \text{and}$$

$$\left\{ \begin{array}{l} \pi_i \alpha_i < \pi_a \alpha_a \ \text{and} \ \pi_a \alpha_a - \pi_i \alpha_i + c_a \leq (1 - \delta)p, \ \text{or} \\[2mm] \pi_i \alpha_i > \pi_a \alpha_a \ \text{and} \ c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i, \ \text{or} \\[2mm] \hspace{4cm} \pi_i \alpha_i = \pi_a \alpha_a \ \text{and} \ (1 - \delta)p - c_a \geq 0 \end{array} \right\}. \tag{A.26}$$

In this case, the threshold for the consumer indifferent between purchasing the automated

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patching option and not purchasing at all, v_a, satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \qquad (A.27)$$

For this to be an equilibrium, we it is necessary and sufficient to have $0 < v_a < 1$, type $v = 1$ prefers (B, AP) over (B, P), and all v prefer (B, AP) over (B, NP). Note that type $v = 1$ preferring (B, AP) over (B, P) implies that $v < 1$ does the same, by (A.23). Also, the type $v = v_a$ preferring (NB, NP) over both (B, NP) and (B, P) implies that all $v < v_a$ do the same, by (A.24).

We always have $v_a > 0$ from our model assumptions, namely $\pi_a \alpha_a < 1$. To have $v_a < 1$, a necessary and sufficient condition is $\delta p + c_a + \pi_a \alpha_a < 1$.

For $v = 1$ to weakly prefer (B, AP) over (B, P), a necessary and sufficient condition is $1 - \delta p - c_a - \pi_a \alpha_a \geq 1 - p - c_p$, which reduces to $(1 - \delta)p \geq \pi_a \alpha_a + c_a - c_p$.

The condition for all v to prefer (B, AP) over (B, NP) depends on the magnitude of $\pi_i \alpha_i$. If $\pi_i \alpha_i < \pi_a \alpha_a$, then if $v = 1$ prefers (B, AP) over (B, NP), then all $v < 1$ do too. Therefore, a necessary and sufficient condition is $v - \delta p - c_a - \pi_a \alpha_a v \geq v - p - (\pi_s \alpha_s u(\sigma) + \pi_i \alpha_i)v$ for $v = 1$. With $u(\sigma) = 0$, this becomes $c_a + p\delta + \pi_a \alpha_a \leq p + \pi_i \alpha_i$. So if $\pi_i \alpha_i < \pi_a \alpha_a$, then we need the condition $c_a + p\delta + \pi_a \alpha_a \leq p + \pi_i \alpha_i$.

On the other hand, if $\pi_i \alpha_i > \pi_a \alpha_a$, then $v = v_a$ preferring (B, AP) over (B, NP) implies that all $v > v_a$ do too. Hence, a necessary and sufficient condition is for $v - \delta p - c_a - \pi_a \alpha_a v \geq v - p - (\pi_s \alpha_s u(\sigma) + \pi_i \alpha_i)v$ for $v = v_a$. This simplifies to $c_a + p(-1 + \delta + \pi_a \alpha_a) \leq (c_a + \delta p)\pi_i \alpha_i$.

In the case of $\pi_i \alpha_i = \pi_a \alpha_a$, we need $(1 - \delta)p - c_a \geq 0$ for everyone to weakly prefer (B, AP) over (B, NP).

We also need $v = v_a$ to weakly prefer (NB, NP) over both (B, NP) and (B, P), so that all $v < v_a$ do too. We need $0 \geq v - p - \pi_i \alpha_i v$ and $0 \geq v - p - c_p$ for $v = v_a$. These simplify to $(c_a + \delta p)\pi_i \alpha_i \geq c_a + p(-1 + \delta + \pi_a \alpha_a)$ and $c_p + (1 - \delta)p \geq c_a + (c_p + p)\pi_a \alpha_a$. Altogether, Case (I) arises if and only if the condition in (A.26) occurs. $\square$

Next, for Case (II), in which there are no consumers choosing (B, P) but the upper tier of consumers is unpatched while the bottom tier chooses automated patching, i.e., $0 < v_a < v_b < 1$, we have $u = 1 - v_b$. Following the same steps as before, we prove the following claim related to the corresponding conditions for which Case (II) arises.

Claim 4 *The equilibrium that corresponds to Case (II) arises if and only if the following conditions are satisfied:*

$$(1 - \delta)p - c_a > 0 \text{ and } (1 - \delta)p < \pi_a \alpha_a - \pi_i \alpha_i + c_a \text{ and } \pi_a \alpha_a - \pi_i \alpha_i + \pi_s \alpha_s > 0 \text{ and}$$

$$\frac{(2\pi_s \alpha_s)(c_a + \delta p)}{1 - \pi_a \alpha_a} < -\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s + \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s(c_a + (\delta - 1)p)} \text{ and}$$

$$\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s \leq \sqrt{(-\pi_a \alpha_a + \pi_i \alpha_i + \pi_s \alpha_s)^2 - 4\pi_s \alpha_s(c_a + (\delta - 1)p)} + 2c_p. \quad (A.28)$$

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, v_a, again satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \qquad (A.29)$$

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The threshold for the consumer indifferent between being unpatched an purchasing the automated patching option, v_b, satisfies

$$v_b = \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s u - \pi_i\alpha_i}. \tag{A.30}$$

Using $u = 1 - v_b$, we find that v_b solves a quadratic equation. To find which root is the solution, we note that v_b must satisfy $\pi_a\alpha_a - \pi_s\alpha_s u - \pi_i\alpha_i > 0$ since higher types choose (B, NP) over (B, AP) by (A.25). This implies $(1-\delta)p - c_a > 0$ in order for $v_b > 0$. Using this, we find that the root of the quadratic which specifies v_b is given by

$$v_b = \frac{\pi_s\alpha_s + \pi_i\alpha_i - \pi_a\alpha_a + \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s - \pi_i\alpha_i)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}}{2\pi_s\alpha_s}. \tag{A.31}$$

For this to be an equilibrium, we need $0 < v_a < v_b < 1$ and type $v = 1$ prefers (B, NP) over (B, P). Note that type $v = 1$ preferring (B, NP) over (B, P) implies that $v < 1$ does the same, by (A.23). This also implies that v_b prefers (B, AP) over (B, P), so that $v < v_b$ also prefer (B, AP) over (B, P), again by (A.23). Moreover, type $v = v_a$ preferring (NB, NP) over both (B, NP) and (B, P) implies that all $v < v_a$ do the same, by (A.24).

Again, we always have $v_a > 0$ from our model assumptions, namely $\pi_a\alpha_a < 1$. For $v_b < 1$, an equivalent condition is $(1-\delta)p < c_a + \pi_a\alpha_a - \pi_i\alpha_i$ and $\pi_i\alpha_i < \pi_s\alpha_s + \pi_a\alpha_a$.

For $v_a < v_b$, it is equivalent for $\sqrt{(\pi_a\alpha_a - \pi_s\alpha_s - \pi_i\alpha_i)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)} > (2\pi_s\alpha_s)\left(\frac{c_a+\delta p}{1-\pi_a\alpha_a}\right) + \pi_a\alpha_a - \pi_s\alpha_s - \pi_i\alpha_i$.

For type $v = 1$ to weakly prefer (B, NP) over (B, P), we equivalently have the condition $\pi_a\alpha_a + \pi_s\alpha_s + \pi_i\alpha_i - 2c_p \leq \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s - \pi_i\alpha_i)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}$. These conditions can all be found in (A.28). □

Next, for case (III), in which all segments are represented and the middle tier is unpatched, i.e., $0 < v_a < v_b < v_p < 1$, we have $u = v_p - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (III) arises.

Claim 5 *The equilibrium that corresponds to case (III) arises if and only if the following conditions are satisfied:*

$$(1-\delta)p - c_a \geq 0 \text{ and } \pi_a\alpha_a > c_p \text{ and } \pi_s\alpha_s(-\pi_a\alpha_a - c_a + c_p - \delta p + p) < (c_p - \pi_a\alpha_a)(c_p - \pi_i\alpha_i) \text{ and}$$
$$\pi_i\alpha_i(c_a + \delta p) + \pi_a\alpha_a c_p > \pi_i\alpha_i(c_p + p) \text{ and}$$
$$\left\{\left(c_a + p(\pi_a\alpha_a + \delta) \leq p\right) \text{ or } \left(c_a + p(\pi_a\alpha_a + \delta) > p \text{ and}\right.\right.$$
$$\pi_s\alpha_s(c_a + \delta p)^2(c_a + \pi_a\alpha_a(c_p + p) - c_p + (\delta - 1)p) +$$
$$\left.\left.(\pi_a\alpha_a - 1)(c_a + p(\pi_a\alpha_a + \delta - 1))(\pi_i\alpha_i(c_a + \delta p) - c_a - p(\pi_a\alpha_a + \delta - 1)) > 0\right)\right\}. \tag{A.32}$$

In this case, the threshold for the consumer indifferent between purchasing the automated

patching option and not purchasing at all, v_a, again satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \tag{A.33}$$

To solve for the thresholds v_b and v_p, using $u = v_p - v_b$, note that they solve

$$v_b = \frac{(1 - \delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s (v_p - v_b) - \pi_i \alpha_i}, \text{ and} \tag{A.34}$$

$$v_p = \frac{c_p}{\pi_s \alpha_s (v_p - v_b) + \pi_i \alpha_i}. \tag{A.35}$$

From (A.34), we have $\pi_s \alpha_s (v_p - v_b) + \pi_i \alpha_i = \pi_a \alpha_a - \frac{(1-\delta)p - c_a}{v_b}$, while from (A.35), we have $\pi_s \alpha_s (v_p - v_b) + \pi_i \alpha_i = \frac{c_p}{v_p}$. Equating these two expressions and solving for v_p in terms of v_b, we have

$$v_p = \frac{c_p v_b}{c_a - (1 - \delta)p + v_b \pi_a \alpha_a}. \tag{A.36}$$

Plugging this expression for v_p into (A.35) and noting that $c_a - (1 - \delta)p + v_b \pi_a \alpha_a > 0$ in order for $v_p > 0$, we find that v_b must be a zero of the cubic equation:

$$f_1(x) \triangleq (c_a - (1-\delta)p + x\pi_a\alpha_a)(c_a - (1-\delta)p + x(\pi_a\alpha_a - \pi_i\alpha_i)) + x^2 \pi_s\alpha_s(c_a - c_p - (1-\delta)p + x\pi_a\alpha_a). \tag{A.37}$$

To find which root of the cubic v_b must be, first note that $\pi_a\alpha_a - \pi_s\alpha_s u - \pi_i\alpha_i > 0$ for consumers of higher valuation to prefer (B, NP) over (B, AP) by (A.25). From that, we have that $(1 - \delta)p - c_a > 0$ in order for $v_b > 0$. To pin down the root of the cubic, note that the cubic's highest order term is $\pi_s\alpha_s\pi_a\alpha_a x^3$, so $\lim_{x \to -\infty} f_1(x) = -\infty$ and $\lim_{x \to \infty} f_1(x) = \infty$. We find $f_1(0) = ((1 - \delta)p - c_a)^2 > 0$ and $f_1\left(\frac{(1-\delta)p - c_a}{\pi_a\alpha_a}\right) = -\frac{c_p \pi_s \alpha_s ((1-\delta)p - c_a)^2}{(\pi_a\alpha_a)^2} < 0$, while $0 < \frac{(1-\delta)p - c_a}{\pi_a\alpha_a}$.

We note that from (A.34), we have that $v_b > \frac{(1-\delta)p - c_a}{\pi_a\alpha_a}$, so it follows that v_b is the largest (i.e., most positive) root of the cubic. Then using (A.36), we solve for v_p.

For this to be an equilibrium, a necessary and sufficient condition is $0 < v_a < v_b < v_p < 1$. This tells us that all $v \in [v_p, 1]$ have the same preferences and will purchase (B, P), all $v \in [v_b, v_p)$ have the same preferences and will purchase (B, NP), and all $v \in [v_a, v_b)$ have the same preferences and will purchase (B, AP). Finally, all $v < v_a$ have the same preferences and will not purchase in equilibrium.

For $v_p < 1$, using (A.36), a necessary and sufficient condition for this to hold is $(1 - \delta)p - c_a < v_b(\pi_a\alpha_a - c_p)$. Since $(1 - \delta)p - c_a > 0$ (again, from $v_b > 0$), we need $\pi_a\alpha_a > c_p$. To have $v_b > \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - c_p}$, a necessary and sufficient condition is that $f_1\left(\frac{(1-\delta)p - c_a}{\pi_a\alpha_a - c_p}\right) < 0$ so that the third root of $f_1(x)$ is greater than $\frac{(1-\delta)p - c_a}{\pi_a\alpha_a - c_p}$. Omitting the algebra, this simplifies to $\pi_s\alpha_s(-\pi_a\alpha_a - c_a + c_p - \delta p + p) < (c_p - \pi_a\alpha_a)(c_p - \pi_i\alpha_i)$.

For $v_b < v_p$, using (A.36), it is equivalent to have $v_b < \frac{(1-\delta)p - c_a + c_p}{\pi_a\alpha_a}$. A necessary and sufficient condition for this is that $f_1\left(\frac{(1-\delta)p - c_a + c_p}{\pi_a\alpha_a}\right) > 0$ so that the third root of $f_1(x)$ is smaller than $\frac{(1-\delta)p - c_a + c_p}{\pi_a\alpha_a}$. This condition becomes $\pi_i\alpha_i(c_a + \delta p) + \pi_a\alpha_a c_p > \pi_i\alpha_i(c_p + p)$.

For $v_a < v_b$, using (A.27), an equivalent condition is $v_b > \frac{\delta p + c_a}{1 - \pi_a\alpha_a}$. Since $v_b > \frac{(1-\delta)p - c_A}{\pi_a\alpha_a}$ (by the

A.18

construction of v_b above as the largest root of the cubic), it follows that if $\frac{(1-\delta)p-c_A}{\pi_a\alpha_a} \geq \frac{\delta p+c_a}{1-\pi_a\alpha_a}$, then we don't need any extra conditions. The condition $\frac{(1-\delta)p-c_A}{\pi_a\alpha_a} \geq \frac{\delta p+c_a}{1-\pi_a\alpha_a}$ simplifies to $(1-\delta-\pi_a\alpha_a)p \geq c_a$. Otherwise, if $(1-\delta-\pi_a\alpha_a)p < c_a$, then we need $f_1\left(\frac{\delta p+c_a}{1-\pi_a\alpha_a}\right) < 0$ for $v_a < v_b$. This condition is $\pi_s\alpha_s(c_a+\delta p)^2(c_a+\pi_a\alpha_a(c_p+p)-c_p+(\delta-1)p)+(\pi_a\alpha_a-1)(c_a+p(\pi_a\alpha_a+\delta-1))(\pi_i\alpha_i(c_a+\delta p)-c_a-p(\pi_a\alpha_a+\delta-1)) > 0$, which is given in (A.32). $\square$

Next, for case (IV), in which all consumers who purchase are unpatched, i.e., $0 < v_b < 1$, we have $u = 1 - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter conditions for which case (IV) arises.

Claim 6 *The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:*

$$1 - 2c_p + \pi_i\alpha_i + \pi_s\alpha_s \leq \sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2} \text{ and } \pi_i\alpha_i < 1 - p \text{ and}$$

$$(1-\pi_a\alpha_a)(1-\pi_i\alpha_i)+(-1+2c_a+2\delta p+\pi_a\alpha_a)\pi_s\alpha_s \geq (1-\pi_a\alpha_a)\sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2} \text{ and}$$

$$\left\{\left(1 + \pi_i\alpha_i + \pi_s\alpha_s - 2\pi_a\alpha_a > 0 \text{ and } p < \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s} \text{ and}\right.\right.$$

$$\left.2(\pi_a\alpha_a+c_a-(1-\delta)p) + \sqrt{(\pi_i\alpha_i+\pi_s\alpha_s-1)^2 + 4p\pi_s\alpha_s} \geq \pi_i\alpha_i + \pi_s\alpha_s + 1\right) \text{ or}$$

$$\left(\left(-2\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s+1 < 0 \text{ or } p > \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s}\right) \text{ and}\right.$$

$$\left.\pi_a\alpha_a(\pi_i\alpha_i+\pi_s\alpha_s)+2\pi_s\alpha_s(c_a+\delta p)+1 \geq \pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s+(1-\pi_a\alpha_a)\sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s}\right) \text{ or}$$

$$\left.\left(p = \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a+\pi_i\alpha_i+\pi_s\alpha_s)}{\pi_s\alpha_s} \text{ and } (1-\delta)p - c_a \leq 0\right)\right\}. \quad \text{(A.38)}$$

To solve for the threshold v_b, using $u = 1 - v_b$, we solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(1 - v_b) - \pi_i\alpha_i}. \quad \text{(A.39)}$$

For this to be an equilibrium, we have that $1 - \pi_s\alpha_s u - \pi_i\alpha_i > 0$, otherwise all consumers would prefer (NB, NP) over (B, NP), which can't happen in equilibrium. Using $1 - \pi_s\alpha_s(1-v_b) - \pi_i\alpha_i > 0$, we find the right root of the quadratic for v_b to be

$$v_b = \frac{1}{2} + \frac{-1 + \pi_i\alpha_i + \sqrt{(1 - \pi_s\alpha_s - \pi_i\alpha_i)^2 + 4p\pi_s\alpha_s}}{2\pi_s\alpha_s}. \quad \text{(A.40)}$$

For this to be an equilibrium, the necessary and sufficient conditions are that $0 < v_b < 1$, type $v = 1$ weakly prefers (B, NP) to both (B, AP) over (B, P), and $v = v_b$ weakly prefers (NB, NP) over (B, AP).

For $0 < v_b < 1$, it is equivalent to have $\pi_i\alpha_i < 1 - p$.

For $v = 1$ to prefer (B, NP) over (B, P), we need $1 - 2c_p + \pi_i\alpha_i + \pi_s\alpha_s \leq \sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$.

For $v = v_b$ to weakly prefer (NB, NP) over (B, AP), we need $0 \geq v_b - \delta p - c_a - \pi_a\alpha_a v_b$. This simplifies to $(1-\pi_a\alpha_a)(1-\pi_i\alpha_i)+(-1+2c_a+2\delta p+\pi_a\alpha_a)\pi_s\alpha_s \geq (1-\pi_a\alpha_a)\sqrt{4p\pi_s\alpha_s + (1 - \pi_s\alpha_s - \pi_i\alpha_i)^2}$.

For everyone to prefer (B, NP) over (B, AP), the condition needed depends on whether $u(\sigma) > \frac{\pi_a\alpha_a - \pi_i\alpha_i}{\pi_s\alpha_s}$, as seen in (A.25). If $u(\sigma) > \frac{\pi_a\alpha_a - \pi_i\alpha_i}{\pi_s\alpha_s}$, then lower valuation consumers would prefer (B, NP) over (B, AP) so that a sufficient condition for everyone to prefer (B, NP) over (B, AP) is that $v = 1$ weakly prefers (B, NP) over (B, AP). On the other hand, if $u(\sigma) < \frac{\pi_a\alpha_a - \pi_i\alpha_i}{\pi_s\alpha_s}$, then higher valuation consumers prefer (B, NP) over (B, AP) so that the condition would be $v = v_b$ weakly prefers (B, NP) over (B, AP).

The condition $u(\sigma) > \frac{\pi_a\alpha_a - \pi_i\alpha_i}{\pi_s\alpha_s}$ is equivalent to $1 + \pi_i\alpha_i + \pi_s\alpha_s - 2\pi_a\alpha_a > 0$ and $p < \frac{(1-\pi_a\alpha_a)(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)}{\pi_s\alpha_s}$.

The condition that $v = 1$ weakly prefers (B, NP) over (B, AP) is $v - p - ((1-v_b)\pi_s\alpha_s v + \pi_i\alpha_i v) \geq v - \delta p - c_a - \pi_a\alpha_a v$ for $v = v_b$. This simplifies to $2(\pi_a\alpha_a + c_a - (1-\delta)p) + \sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s} \geq \pi_i\alpha_i + \pi_s\alpha_s + 1$.

The condition that $v = v_b$ weakly prefers (B, NP) over (B, AP) is $\pi_a\alpha_a(\pi_i\alpha_i + \pi_s\alpha_s) + 2\pi_s\alpha_s(c_a + \delta p) + 1 \geq \pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + (1 - \pi_a\alpha_a)\sqrt{(\pi_i\alpha_i + \pi_s\alpha_s - 1)^2 + 4p\pi_s\alpha_s}$.

Lastly, if $u(\sigma) = \frac{\pi_a\alpha_a - \pi_i\alpha_i}{\pi_s\alpha_s}$, then everyone will prefer (B, NP) over (B, AP) if $(1-\delta)p - c_a \leq 0$. The conditions of these subcases are given in (A.38).

Next, for case (V), in which the lower tier of purchasing consumers is unpatched while the upper tier does automated patching, i.e., $0 < v_b < v_a < 1$, we have $u = v_a - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (V) arises.

Claim 7 *The equilibrium that corresponds to case (V) arises if and only if the following conditions are satisfied:*

$$(1-\delta)p - c_a < 0 \ \text{ and } \ (\pi_a\alpha_a + c_a - 1 - (1-\delta)p)(\pi_a\alpha_a - \pi_i\alpha_i + c_a - (1-\delta)p) > \pi_s\alpha_s(\pi_a\alpha_a + c_a + \delta p - 1) \ \text{ and}$$
$$\pi_i\alpha_i(c_a + \delta p) < c_a + p(-1 + \delta + \pi_a\alpha_a) \ \text{ and } \ \pi_a\alpha_a \leq (1 - \delta)p + c_p - c_a. \quad \text{(A.41)}$$

To solve for the thresholds v_b and v_a, using $u = v_a - v_b$, note that they solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(v_a - v_b) - \pi_i\alpha_i}, \ \text{ and} \quad \text{(A.42)}$$

$$v_a = \frac{(1 - \delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s(v_a - v_b) - \pi_i\alpha_i}. \quad \text{(A.43)}$$

From (A.42), we have $\pi_s\alpha_s(v_a - v_b) + \pi_i\alpha_i = 1 - \frac{p}{v_b}$, while from (A.43), we have $\pi_s\alpha_s(v_a - v_b) + \pi_i\alpha_i = \pi_a\alpha_a - \frac{(1-\delta)p - c_a}{v_a}$. Equating these two expressions and solving for v_a in terms of v_b, we have

$$v_a = \frac{v_b(-c_a + (1 - \delta)p)}{p - v_b(1 - \pi_a\alpha_a)}. \quad \text{(A.44)}$$

Plugging this expression for v_a into (A.42), we find that v_b must be a zero of the cubic equation:

$$f_2(x) \triangleq (1 - \pi_a\alpha_a)\pi_s\alpha_s x^3 + ((1 - \pi_a\alpha_a)(1 - \pi_i\alpha_i) - c_a\pi_s\alpha_s - \delta p\pi_s\alpha_s)x^2 - p(2 - \pi_a\alpha_a - \pi_i\alpha_i)x + p^2.$$
$$\text{(A.45)}$$

A.20

To find which root of the cubic v_b must be, first note that $\pi_a \alpha_a - \pi_s \alpha_s u - \pi_i \alpha_i < 0$ for consumers of higher valuation to prefer (B, NP) over (B, AP) by (A.25). From that, we have that $c_a - (1-\delta)p > 0$ in order for $v_a > 0$. To pin down the root of the cubic, note that the cubic's highest order term is $\pi_s \alpha_s (1 - \pi_a \alpha_a) x^3$, so $\lim_{x \to -\infty} f_2(x) = -\infty$ and $\lim_{x \to \infty} f_2(x) = \infty$. We find $f_2(0) = p^2 > 0$ and $f_2\left(\frac{p}{1-\pi_a \alpha_a}\right) = -\frac{p^2 \pi_s \alpha_s (c_a - (1-\delta)p)}{(1 - \pi_a \alpha_a)^2} < 0$. Since $\lim_{x \to \infty} f_2(x) = \infty$, there exists a root larger than $\frac{p}{1-\pi_a \alpha_a}$. Note that from (A.42), we have $v_b > \frac{p}{1-\pi_a \alpha_a}$. Therefore, v_b is the largest root of the cubic, lying past $\frac{p}{1-\pi_a \alpha_a}$. Then using (A.44), we solve for v_a.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_a < 1$ and type $v = 1$ prefers (B, AP) over (B, P). Type $v = 1$ preferring (B, AP) over (B, P) ensures $v < 1$ does so too, by (A.23). Moreover, since type $v = v_a$ is indifferent between (B, AP) and (B, NP), and since (B, AP) is preferred over (B, P), by transitivity, it follows that type v_a prefers (B, NP) over (B, P). It follows that $v < v_a$ prefers (B, NP) over (B, P) as well by (A.23).

For $v_a < 1$, using (A.44), an equivalent condition for this to hold is $p + v_b(-1 + c_a - p(1 - \delta) + \pi_a \alpha_a) < 0$. If $c_a - p(1 - \delta) \geq 1 - \pi_a \alpha_a$, then this case can't happen. Otherwise, if $c_a - p(1-\delta) < 1 - \pi_a \alpha_a$, then this condition becomes $v_b > \frac{p}{1 - c_a + p(1-\delta) - \pi_a \alpha_a}$. This is equivalent to $f_2(\frac{p}{1 - c_a + p(1-\delta) - \pi_a \alpha_a}) > 0$, which simplifies to $(\pi_a \alpha_a + c_a - 1 - (1-\delta)p)(\pi_a \alpha_a - \pi_i \alpha_i + c_a - (1-\delta)p) > \pi_s \alpha_s (\pi_a \alpha_a + c_a + \delta p - 1)$.

For $v_a > v_b$, using (A.44), it is equivalent to require $v_b < \frac{c_a + p\delta}{1 - \pi_a \alpha_a}$. For this to happen, we need the condition $f_2(\frac{c_a + p\delta}{1 - \pi_a \alpha_a}) > 0$, which simplifies to $\pi_i \alpha_i (c_a + \delta p) < c_a + p(-1 + \delta + \pi_a \alpha_a)$.

For $v_b > 0$, this holds by construction of v_b as the largest root of $f_2(x)$ (which was shown to be larger than $\frac{p}{1-\pi_a \alpha_a} > 0$), so no additional conditions are needed.

Finally, for type $v = 1$ to prefer (B, AP) over (B, P), a necessary and sufficient condition is $\pi_a \alpha_a \leq (1-\delta)p + c_p - c_a$. The conditions above are summarized in (A.41). $\square$

Next, for case (VI), in which all segments are represented and the middle tier does automated patching, i.e., $0 < v_b < v_a < v_p < 1$, we have $u = v_a - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VI) arises.

Claim 8 *The equilibrium that corresponds to case (VI) arises if and only if the following conditions are satisfied:*

$$(1-\delta)p - c_a < 0 \text{ and } c_p + (1-\delta)p < c_a + \pi_a \alpha_a \text{ and } c_p(1 - \pi_a \alpha_a) + (1-\delta)p > c_a \text{ and}$$
$$c_p(\pi_a \alpha_a)^2(-c_a + (1-\delta)p + c_p(1 - \pi_a \alpha_a)) + \pi_s \alpha_s(-c_a + (1-\delta)p + c_p)^2(c_a - c_p(1 - \pi_a \alpha_a) + (\pi_a \alpha_a - 1 + \delta)p) -$$
$$(c_a - c_p - (1-\delta)p)(c_a - (1-\delta)p - c_p(1 - \pi_a \alpha_a))\pi_a \alpha_a \pi_i \alpha_i < 0 \text{ and}$$
$$c_a + p(-1 + \delta + \pi_a \alpha_a) > (c_a + \delta p)\pi_i \alpha_i. \quad \text{(A.46)}$$

To solve for the thresholds v_b and v_a, using $u = v_a - v_b$, note that they solve

$$v_b = \frac{p}{1 - \pi_s \alpha_s(v_a - v_b) - \pi_i \alpha_i}, \text{ and} \qquad (A.47)$$

$$v_a = \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s(v_a - v_b) - \pi_i \alpha_i}. \qquad (A.48)$$

These are the same as (A.42) and (A.43). Using the exact same argument, it follows that v_b

is the largest root of the cubic $f_2(x)$, lying past $\frac{c_a-(1-\delta)p}{1-\pi_a\alpha_a}$. Note that the largest root v_b is indeed larger $\frac{p}{1-\pi_a\alpha_a}$ in this case as well since $v_b = \frac{pv_a}{-c_a+p(1-\delta)+v_a(1-\pi_a\alpha_a)}$ and $(1-\delta)p - c_a < 0$. Then using (A.44), we solve for v_a.

In this case, however, we also have a standard patching population, with the standard patching threshold given by $v_p = \frac{c_p-(c_a-(1-\delta)p)}{\pi_a\alpha_a}$.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_a < v_p < 1$. This tells us that all $v \in [v_p, 1]$ have the same preferences and will purchase (B, P), all $v \in [v_a, v_p)$ have the same preferences and will purchase (B, AP), and all $v \in [v_b, v_a)$ have the same preferences and will purchase (B, NP). Finally, all $v < v_b$ have the same preferences and will not purchase in equilibrium.

For $v_p < 1$, the necessary and sufficient condition is $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$.

For $v_a < v_p$, using (A.44) to write v_a in terms of v_b, it is equivalent to write $v_b((1-\delta)p - c_a + c_p(1 - \pi_a\alpha_a)) > p((1-\delta)p - c_a + c_p)$.

If $-c_a + c_p(1 - \pi_a\alpha_a) + p(1-\delta) > 0$, then we can rewrite this as $v_b > \frac{p((1-\delta)p-c_a+c_p)}{(1-\delta)p-c_a+c_p(1-\pi_a\alpha_a)}$. This is equivalent to $f_2(\frac{p((1-\delta)p-c_a+c_p)}{(1-\delta)p-c_a+c_p(1-\pi_a\alpha_a)}) < 0$, which simplifies to $c_p(\pi_a\alpha_a)^2(-c_a + (1-\delta)p + c_p(1 - \pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + (1-\delta)p + c_p)^2(c_a - c_p(1 - \pi_a\alpha_a) + (\pi_a\alpha_a - 1 + \delta)p) - (c_a - c_p - (1-\delta)p)(c_a - (1-\delta)p - c_p(1 - \pi_a\alpha_a))\pi_a\alpha_a\pi_i\alpha_i < 0$.

On the other hand, if $-c_a + c_p(1 - \pi_a\alpha_a) + p(1-\delta) < 0$, then we need $v_b < \frac{p((1-\delta)p-c_a+c_p)}{(1-\delta)p-c_a+c_p(1-\pi_a\alpha_a)}$. If $p((1-\delta)p - c_a + c_p) \geq 0$, then this can't happen since the denominator is negative and $v_b > 0$. If $p((1-\delta)p - c_a + c_p) < 0$, then $\frac{p((1-\delta)p-c_a+c_p)}{(1-\delta)p-c_a+c_p(1-\pi_a\alpha_a)} < p$. However, $v_b > \frac{p}{1-\pi_a\alpha_a}$, so this can't happen either. Therefore, $-c_a + c_p(1 - \pi_a\alpha_a) + p(1-\delta) < 0$ rules out this case.

Lastly, if $-c_a + c_p(1 - \pi_a\alpha_a) + p(1-\delta) = 0$, then this $v_b((1-\delta)p - c_a + c_p(1 - \pi_a\alpha_a)) > p((1-\delta)p - c_a + c_p)$ becomes $0 > ((1-\delta)p - c_a + c_p)$. This simplifies to $0 < c_p\pi_a\alpha_a$, which can't happen.

Therefore, the conditions for $v_a < v_p$ are $c_p(1 - \pi_a\alpha_a) + (1-\delta)p > c_a$ and $c_p(\pi_a\alpha_a)^2(-c_a + (1-\delta)p + c_p(1 - \pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + (1-\delta)p + c_p)^2(c_a - c_p(1 - \pi_a\alpha_a) + (\pi_a\alpha_a - 1 + \delta)p) - (c_a - c_p - (1-\delta)p)(c_a - (1-\delta)p - c_p(1 - \pi_a\alpha_a))\pi_a\alpha_a\pi_i\alpha_i < 0$.

For $v_b < v_a$, again using (A.44) to write v_a in terms of v_b, this simplifies to $v_b < \frac{c_a+p\delta}{1-\pi_a\alpha_a}$. This can equivalently be expressed as $f_2\left(\frac{c_a+p\delta}{1-\pi_a\alpha_a}\right) > 0$, which simplifies to $c_a + p(-1 + \delta + \pi_a\alpha_a) > (c_a + \delta p)\pi_i\alpha_i$.

For $0 < v_b$, no condition is needed since v_b is defined to be the largest root of the cubic, which was shown to be larger than $\frac{p}{1-\pi_a\alpha_a}$.

A summary of the above necessary and sufficient conditions is given in (A.46). □

Next, for case (VII), in which there are no automated patching users while the lower tier is unpatched and the upper tier is patched, i.e., $0 < v_b < v_p < 1$, we have $u = v_p - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VII) arises.

Claim 9 *The equilibrium that corresponds to case (VII) arises if and only if the following condi-*

tions are satisfied:

$$(-1 + c_p + p)\pi_s\alpha_s < (1 - c_p)(-c_p + \pi_i\alpha_i) \text{ and } \pi_i\alpha_i < \frac{c_p}{c_p + p} \text{ and}$$

$$(1 - \pi_a\alpha_a)(c_a + (\pi_a\alpha_a - (1 - \delta))p)(c_a + p(\pi_a\alpha_a - (1 - \delta)) - (c_a + \delta p)\pi_i\alpha_i) +$$
$$(c_a + \delta p)^2(c_a - c_p - (1 - \delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s \geq 0 \text{ and } c_a + p(\delta + \pi_a\alpha_a) > p \text{ and}$$

$$\left\{ \left(c_a - (1 - \delta)p + c_p(-1 + \pi_a\alpha_a) \geq 0 \right) \text{ or} \right.$$

$$\left(c_a + c_p(-1 + \pi_a\alpha_a) < 0 \text{ and } \pi_a\alpha_a(c_a - (1 - \delta)p + c_p(-1 + \pi_a\alpha_a))(c_p\pi_a\alpha_a + (c_a - c_p - (1 - \delta)p)\pi_i\alpha_i) \leq \right.$$

$$\left. \left. (-c_a + c_p + (1 - \delta)p)^2(c_a - c_p - (1 - \delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s \right) \right\}. \quad \text{(A.49)}$$

To solve for the thresholds v_b and v_p, using $u = v_p - v_b$, note that they solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(v_p - v_b) - \pi_i\alpha_i}, \text{ and} \quad \text{(A.50)}$$

$$v_p = \frac{c_p}{\pi_s\alpha_s(v_p - v_b) + \pi_i\alpha_i}. \quad \text{(A.51)}$$

From (A.50), we have $\pi_s\alpha_s(v_p - v_b) + \pi_i\alpha_i = 1 - \frac{p}{v_b}$, while from (A.51), we have $\pi_s\alpha_s(v_p - v_b) + \pi_i\alpha_i = \frac{c_p}{v_p}$. Equating these two expressions and solving for v_p in terms of v_b, we have

$$v_p = \frac{c_p v_b}{v_b - p}. \quad \text{(A.52)}$$

Plugging this expression for v_p into (A.50), we find that v_b must be a zero of the cubic equation:

$$f_3(x) \triangleq \pi_s\alpha_s x^3 + (1 - \pi_i\alpha_i - \pi_s\alpha_s(c_p + p))x^2 - (2 - \pi_i\alpha_i)px + p^2. \quad \text{(A.53)}$$

To find which root of the cubic v_b must be, note that the cubic's highest order term is $\pi_s\alpha_s x^3$, so $\lim_{x \to -\infty} f_3(x) = -\infty$ and $\lim_{x \to \infty} f_3(x) = \infty$. We find $f_3(0) = p^2 > 0$, and $f_3(p) = -c_p\pi_s\alpha_s p^2 < 0$. Since $v_b - p > 0$ in equilibrium, we have that v_b is the largest root of the cubic, lying past p. Then using (A.52), we solve for v_p.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_p < 1$, type $v = v_p$ prefers (B, P) over (B, AP), and type $v = v_b$ prefers (NB, NP) to (B, AP). Type $v = v_p$ preferring (B, P) over (B, AP) ensures $v > v_p$ also prefer (B, P) over (B, AP), by (A.23). Moreover, type $v = v_b$ preferring (NB, NP) over (B, AP) ensures $v < v_b$ do so too, by (A.24).

For $v_p < 1$, a necessary and sufficient condition for this to hold is $v_b > \frac{p}{1-c_p}$. This is equivalent to $f_3(\frac{p}{1-c_p}) < 0$. This simplifies to $(-1 + c_p + p)\pi_s\alpha_s < (1 - c_p)(-c_p + \pi_i\alpha_i)$.

For $v_p > v_b$, a necessary and sufficient condition is $v_b < c_p + p$. This is equivalent to $f_3(c_p + p) > 0$, or $\pi_i\alpha_i < \frac{c_p}{c_p + p}$.

We don't need any conditions for $v_b > 0$, since by construction, $v_b > p > 0$.

For type $v = v_p$ to prefer (B, P) over (B, AP), a necessary and sufficient condition is $\frac{c_p v_b}{v_b - p} \geq \frac{c_p - (c_a - (1-\delta)p)}{\pi_a\alpha_a}$. This simplifies to $v_b(c_a - (1-\delta)p - c_p(1 - \pi_a\alpha_a)) \geq p(c_a - (1-\delta)p - c_p)$. This can be

A.23

broken down into three cases, depending on the sign of $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a)$ (also considering the case when the factor is zero). When $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) = 0$, the left side is 0 while the right side is negative, so this inequality holds. If $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) > 0$, then the inequality becomes $v_b \geq p\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}$. But $\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)} < 1$, and since $v_b > p$ by construction, this inequality holds without further conditions. On the other hand, if $c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a) < 0$, then we need $v_b \leq p\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}$. So we need $f_3\left(\frac{p(c_a-(1-\delta)p-c_p)}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}\right) \geq 0$. Omitting the algebra, this simplifies to $\pi_a\alpha_a(c_a - (1-\delta)p + c_p(-1+\pi_a\alpha_a))(c_p\pi_a\alpha_a + (c_a - c_p - (1-\delta)p)\pi_i\alpha_i) \leq (-c_a + c_p + (1-\delta)p)^2(c_a - c_p - (1-\delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s$.

For $v = v_b$ to prefer (NB, NP) to (B, AP), a necessary and sufficient condition is $v_b \leq \frac{\delta p + c_a}{1-\pi_a\alpha_a}$, which becomes $f_3\left(\frac{\delta p + c_a}{1-\pi_a\alpha_a}\right) \geq 0$. This simplifies to $c_a > (1-\delta-\pi_a\alpha_a)p$ and $(1-\pi_a\alpha_a)(c_a + (\pi_a\alpha_a - (1-\delta))p)(c_a + p(\pi_a\alpha_a - (1-\delta))) - (c_a + \delta p)\pi_i\alpha_i) + (c_a + \delta p)^2(c_a - c_p - (1-\delta)p + (c_p + p)\pi_a\alpha_a)\pi_s\alpha_s \geq 0$. The conditions are summarized in (A.49). $\square$

Next, for case (VIII), in which there are no unpatched users while the lower tier chooses automated patching and the upper tier chooses standard patching, i.e., $0 < v_a < v_p < 1$, we have $u = 0$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VIII) arises.

Claim 10 *The equilibrium that corresponds to case (VIII) arises if and only if the following conditions are satisfied:*

$$c_p+(1-\delta)p < c_a+\pi_a\alpha_a \text{ and } c_p+(1-\delta)p > c_a+(c_p+p)\pi_a\alpha_a \text{ and } (c_p-c_a+(1-\delta)p)\pi_i\alpha_i \geq c_p\pi_a\alpha_a \text{ and}$$

$$\left\{\left(\pi_i\alpha_i < \pi_a\alpha_a \text{ and } c_p\pi_a\alpha_a + \pi_i\alpha_i(c_a - c_p - (1-\delta)p) \leq 0\right) \text{ or}\right.$$

$$\left(\pi_i\alpha_i > \pi_a\alpha_a \text{ and } c_a + p(-1+\delta+\pi_a\alpha_a) \leq (c_a + \delta p)\pi_i\alpha_i\right) \text{ or}$$

$$\left.\left(\pi_i\alpha_i = \pi_i\alpha_i \text{ and } (1-\delta)p - c_a \geq 0\right)\right\}. \quad (A.54)$$

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, v_a, satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}, \quad (A.55)$$

and the consumer indifferent between choosing automated patching and standard patching is given by

$$v_p = \frac{(1-\delta)p + c_p - c_a}{\pi_a\alpha_a}, \quad (A.56)$$

For this to be an equilibrium, it is necessary and sufficient to have $0 < v_a < v_p < 1$, no one prefers (B, NP) over (B, AP), and no one prefers (B, NP) over (B, P).

For $v_p < 1$, this is equivalently written as $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$.

To have $v_a < v_p$, a necessary and sufficient condition is $c_p + (1-\delta)p > c_a + (c_p + p)\pi_a\alpha_a$.

We always have $v_a > 0$ from our model assumptions, namely $\pi_a\alpha_a < 1$.

To ensure that no one prefers (B, NP) over (B, P), it suffices to make $v = v_p$ weakly prefer

(B, P) over (B, NP) so that everyone of higher valuation would also have the same preference (by (A.23)). This condition then becomes $(c_p - c_a + (1 - \delta)p)\pi_i\alpha_i \geq c_p\pi_a\alpha_a$.

To ensure that no one prefers (B, NP) over (B, AP), we need $(\pi_a\alpha_a - (\pi_s\alpha_s u(\sigma) + \pi_i\alpha_i))v \leq (1 - \delta)p - c_a$. In this case, $u(\sigma) = 0$ so that there are three cases, depending on the sign of $\pi_a\alpha_a - \pi_i\alpha_i$.

If $\pi_a\alpha_a > \pi_i\alpha_i$, then higher valuation consumers prefer (B, NP) over (B, AP), so a necessary and sufficient condition is for $v = v_p$ to weakly prefer (B, AP) over (B, NP). This becomes $c_p\pi_a\alpha_a + \pi_i\alpha_i(c_a - c_p - (1 - \delta)p) \leq 0$.

On the other hand, if $\pi_a\alpha_a < \pi_i\alpha_i$, then lower valuation consumers prefer (B, NP) over (B, AP). In this case, a necessary and sufficient for no one to prefer (B, NP) over (B, AP) is for $v = v_a$ to weakly prefer (B, AP) over (B, NP). This simplifies to $c_a + p(-1 + \delta + \pi_a\alpha_a) \leq (c_a + \delta p)\pi_i\alpha_i$.

Lastly, if $\pi_a\alpha_a = \pi_i\alpha_i$, then we need $(1 - \delta)p - c_a \geq 0$ for everyone to prefer (B, AP) over (B, NP). Altogether, Case (VIII) arises if and only if the condition in (A.54) occurs. $\square$

Lastly, for case (IX), in which all users choose standard patching, i.e., $0 < v_p < 1$, we have $u = 0$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (IX) arises.

Claim 11 *The equilibrium that corresponds to case (IX) arises if and only if the following conditions are satisfied:*

$$c_p + p < 1 \ and \ c_a + (c_p + p)\pi_a\alpha_a \geq c_p + (1 - \delta)p \ and \ (c_p + p)\pi_i\alpha_i \geq c_p. \tag{A.57}$$

In this case, the threshold for the consumer indifferent between choosing the standard patching option and not purchasing at all, v_p, satisfies

$$v_p = c_p + p. \tag{A.58}$$

For this to be an equilibrium, it is necessary and sufficient to have $0 < v_p < 1$, $v = v_p$ prefers (NB, NP) over both (B, NP) and (B, AP), and $v = v_p$ prefers (B, P) over both (B, NP) and (B, AP).

For $v_p < 1$, this is equivalently written as $c_p + p < 1$.

For $v = v_p$ to weakly prefer (NB, NP) over (B, NP), we need $0 \geq v_p - p - \pi_i\alpha_i v_p$, which simplifies to $(c_p + p)\pi_i\alpha_i \geq c_p$.

For $v = v_p$ to weakly prefer (NB, NP) over (B, AP), we need $c_a + (c_p + p)\pi_a\alpha_a \geq c_p + (1 - \delta)p$.

For $v = v_p$ to weakly prefer (B, P) over (B, NP) and (B, AP), the conditions will be the same as above since $v = v_p$ is indifferent between (NB, NP) and (B, P).

Altogether, Case (IX) arises if and only if the condition in (A.57) occurs.

This completes the proof of the general consumer market equilibrium for the proprietary case. $\square$

Proof of Lemma 5: Technically, we prove that there exists an $\tilde{\alpha}_2$ such that if $\pi_i\alpha_i < \min\left[\frac{c_p\pi_a\alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$, then for $\alpha_s > \tilde{\alpha}_2$, p^* and δ^* are set so that

1. if $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing,

A.25

2. if $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and if

3. if $c_a > c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

The sketch of the proof is similar to that of Lemma 4. Using Lemma A.4 instead of Lemma A.3, we find the profit-maximizing price within the compact closure of each case, so that the price that induces the largest profit among the cases will be the equilibrium price set by the vendor.

The conditions of this lemma precludes certain candidate market structures from arising in equilibrium. Specifically, using Lemma A.4, $\pi_i\alpha_i < \min\left[\pi_a\alpha_a, \frac{c_p}{1+c_p}\right]$ rules out Cases (VIII) and (IX). We consider the remaining consumer structures that can arise when the vendor sets prices optimally.

Suppose $0 < v_a < 1$ is induced. By part (I) of Lemma A.4, we obtain $v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}$. The profit function in this case is $\Pi_I(p, \delta) = \delta p(1 - v_a(p, \delta))$. Let C_I be the compact closure of the region of the parameter space defining $0 < v_a < 1$, given in part (I) of Lemma A.4. By the Weierstrass extreme value theorem, there exists p and δ in C_I that maximizes $\Pi(p, \delta)$. This p and δ combination may be on the boundary, and we show that the vendor's profit function is continuous across region boundaries later. Otherwise, if this p and δ are interior, the unconstrained maximizer satisfies the first-order conditions.

Differentiating the profit function with respect to p, we have that $p_I^*(\delta) = \frac{1 - c_a - \pi_a\alpha_a}{2\delta}$. The second-order condition gives $\frac{\partial^2}{\partial p^2}\Pi(p, \delta) = -\frac{2\delta^2}{1 - \pi_a\alpha_a}$. We see that

$$\Pi_I \triangleq \Pi(p^*(\delta), \delta) = \frac{(1 - c_a - \pi_a\alpha_a)^2}{4(1 - \pi_a\alpha_a)} \tag{A.59}$$

for any δ, so this is the maximal profit of this case.

On the other hand, suppose $0 < v_a < v_b < 1$ is induced. By part (II) of Lemma A.4, we obtain that $v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}$ and $v_b = \frac{-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + \sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta-1)p)}}{2\pi_s\alpha_s}$. The profit function in this case is

$$\Pi_{II}(p, \delta) = p(1 - v_b(p, \delta)) + \delta p(v_b(p, \delta) - v_a(p, \delta)).$$

The first-order condition in p yields

$$\frac{(\delta - 1)p}{\sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta-1)p)}} +$$

$$\delta\left(\frac{-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + \sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta-1)p)}}{2\pi_s\alpha_s} - \frac{c_a + \delta p}{1 - \pi_a\alpha_a}\right) +$$

$$\delta p\left(-\frac{\delta}{1 - \pi_a\alpha_a} - \frac{\delta - 1}{\sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta-1)p)}}\right) -$$

$$\frac{-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s + \sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta-1)p)}}{2\pi_s\alpha_s} + 1 = 0. \tag{A.60}$$

Letting $X = \sqrt{(-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2 - 4\pi_s\alpha_s(c_a + (\delta - 1)p)}$, we can rewrite this as

$$X^2 + \left(-\pi_a\alpha_a + \pi_i\alpha_i + \frac{\pi_s\alpha_s(\pi_a\alpha_a + \delta(\pi_a\alpha_a + 2c_a + 4\delta p - 1) - 1)}{(\delta - 1)(\pi_a\alpha_a - 1)}\right)X + 2\pi_s\alpha_s(1-\delta)p = 0. \quad \text{(A.61)}$$

Similarly, the first-order condition in δ can be written as

$$X^2 + \left(-\pi_a\alpha_a + \pi_i\alpha_i + \frac{\pi_s\alpha_s(\pi_a\alpha_a + 2c_a + 4\delta p - 1)}{\pi_a\alpha_a - 1}\right)X + 2\pi_s\alpha_s(1 - \delta)p = 0. \quad \text{(A.62)}$$

Using the first-order conditions together, we have that $X(1 - c_a - 2p\delta - \pi_a\alpha_a) = 0$. If $X = 0$, then using the definition of X, we have that $(1 - \delta)p = \frac{4c_a\pi_s\alpha_s - (-\pi_a\alpha_a + \pi_i\alpha_i + \pi_s\alpha_s)^2}{4\pi_s\alpha_s}$. However, if $\pi_s\alpha_s > \pi_a\alpha_a - \pi_i\alpha_i + 2\sqrt{(c_a + 1)(\pi_a\alpha_a - \pi_i\alpha_i + c_a + 1)} + 2c_a + 2$, then $\delta p > 1 + p$. This can't happen in equilibrium since $\delta p < 1$ for consumers to be willing to pay for the automated patching option. Therefore, it cannot be the case that $X = 0$.

Then from $X(1 - c_a - 2p\delta - \pi_a\alpha_a) = 0$, we have that $\delta^*(p) = \frac{1 - c_a - \pi_a\alpha_a}{2p}$. Plugging this back into the profit function and maximizing over p again, we have two roots for p:

$$p = -\left(\frac{1}{18\pi_s\alpha_s}\right)\Bigg(\pi_s\alpha_s(\pi_a\alpha_a + 8\pi_i\alpha_i) + 2(\pi_a\alpha_a - \pi_i\alpha_i)^2 + 2(\pi_s\alpha_s)^2 - 3(c_a + 3)\pi_s\alpha_s \mp$$

$$2\sqrt{(\pi_a\alpha_a - \pi_i\alpha_i + \pi_s\alpha_s)^2\left((\pi_a\alpha_a - \pi_i\alpha_i)^2 + (\pi_s\alpha_s)^2 + \pi_s\alpha_s(-\pi_a\alpha_a + \pi_i\alpha_i - 3c_a)\right)}\Bigg) \quad \text{(A.63)}$$

However, when

$$\pi_s\alpha_s > \frac{1}{8(1 - \pi_i\alpha_i)}\Bigg((\pi_a\alpha_a)^2 - 2\pi_a\alpha_a + 8(\pi_i\alpha_i)^2 + c_a^2 + 2c_a\pi_a\alpha_a - 8c_a\pi_i\alpha_i + 6c_a + 9 -$$

$$16\pi_i\alpha_i - (\pi_a\alpha_a - 4\pi_i\alpha_i + c_a + 3)\sqrt{(\pi_a\alpha_a - 1)(\pi_a\alpha_a + 8\pi_i\alpha_i - 9) + c_a^2 + 2c_a(\pi_a\alpha_a - 4\pi_i\alpha_i + 3)},\Bigg)$$

$$\text{(A.64)}$$

then the smaller root will be negative while the larger root is positive. Therefore, the equilibrium price of this case is

$$p_{II}^* = -\left(\frac{1}{18\pi_s\alpha_s}\right)\Bigg(\pi_s\alpha_s(\pi_a\alpha_a + 8\pi_i\alpha_i) + 2(\pi_a\alpha_a - \pi_i\alpha_i)^2 + 2(\pi_s\alpha_s)^2 - 3(c_a + 3)\pi_s\alpha_s -$$

$$2\sqrt{(\pi_a\alpha_a - \pi_i\alpha_i + \pi_s\alpha_s)^2\left((\pi_a\alpha_a - \pi_i\alpha_i)^2 + (\pi_s\alpha_s)^2 + \pi_s\alpha_s(-\pi_a\alpha_a + \pi_i\alpha_i - 3c_a)\right)}\Bigg). \quad \text{(A.65)}$$

The equilibrium discount of this case is given as

$$\delta_{II}^* = (9\pi_s\alpha_s(\pi_a\alpha_a + c_a - 1))\Bigg(\pi_s\alpha_s(\pi_a\alpha_a + 8\pi_i\alpha_i) + 2(\pi_a\alpha_a - \pi_i\alpha_i)^2 + 2(\pi_s\alpha_s)^2 - 3\pi_s\alpha_s(c_a + 3) -$$

$$2\sqrt{(\pi_a\alpha_a - \pi_i\alpha_i + \pi_s\alpha_s)^2\left((\pi_a\alpha_a - \pi_i\alpha_i)^2 + (\pi_s\alpha_s)^2 + \pi_s\alpha_s(-\pi_a\alpha_a + \pi_i\alpha_i - 3c_a)\right)}\Bigg)^{-1}. \quad \text{(A.66)}$$

The equilibrium profit is given as $\Pi_{II}^* = \Pi_{II}(p_{II}^*, \delta_{II}^*)$. As we had done in Lemma 4, we can

characterize the profit of this case using Taylor series. In particular, there exists an α_6 such for that $\alpha_s > \alpha_6$, the maximal profit of this case is

$$\Pi_{II}^* = \frac{(1 - c_a - \pi_a\alpha_a)^2}{4(1 - \pi_a\alpha_a)} + \frac{(c_a + \pi_a\alpha_a - \pi_i\alpha_i)^2}{4\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k \tag{A.67}$$

for some coefficients c_k. As done before in Lemma 4, we will use a_k, b_k, c_k, and d_k to denote coefficients in the Taylor expansions without referring to specific expressions. These will be used across different cases, and they don't refer to the same quantities or expressions across cases.

On the other hand, suppose $0 < v_a < v_b < v_p < 1$ is induced. By part (III) of Lemma A.4, we obtain that $v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}$, v_b is the most positive root of the cubic $f_3(x) \triangleq \pi_s\alpha_s x^2(c_a - c_p + (\delta - 1)p + \pi_a\alpha_a x) + (c_a + (\delta - 1)p + \pi_a\alpha_a x)(c_a + (\delta - 1)p + x(\pi_a\alpha_a - \pi_i\alpha_i))$, and $v_p = \frac{c_p v_b}{c_a - (1-\delta)p + \pi_a\alpha_a v_b}$. The profit function in this case is

$$\Pi_{III}(p, \delta) = p(1 - v_b(p, \delta)) + \delta p(v_b(p, \delta) - v_a(p, \delta)). \tag{A.68}$$

As we had done in Lemma 4, we employ asymptotic analysis to characterize the equilibrium prices and profit of this case. In particular, since v_b is the most positive root of the cubic equation $f_3(x) \triangleq \pi_s\alpha_s x^2(c_a - c_p + (\delta - 1)p + \pi_a\alpha_a x) + (c_a + (\delta - 1)p + \pi_a\alpha_a x)(c_a + (\delta - 1)p + x(\pi_a\alpha_a - \pi_i\alpha_i))$ (and since $f'(x) \neq 0$ at the value of x that defines v_b), it follows that v_b is an analytic function of the parameters. Letting $v_b = A_0 + \sum_{k=1}^{\infty} d_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$, the cubic equation defining v_b becomes $A_0^2(c_a - c_p - p(1 - \delta) + A_0\pi_a\alpha_a)\pi_s\alpha_s + \sum_{k=0}^{\infty} e_k \left(\frac{1}{\pi_s\alpha_s}\right)^k = 0$ for some coefficients e_k. For this equation to hold, we must have $A_0 = 0$ or $A_0 = \frac{c_p - c_a + (1-\delta)p}{\pi_a\alpha_a}$. The double root $A_0 = 0$ corresponds to the two solutions of the cubic converging to zero, while $A_0 = \frac{c_p - c_a + (1-\delta)p}{\pi_a\alpha_a} > 0$ corresponds to the largest root of cubic.

Then substituting $v_b = \frac{c_p - c_a + (1-\delta)p}{\pi_a\alpha_a} + \frac{A_1}{\pi_s\alpha_s} + \sum_{k=2}^{\infty} d_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$ into $f_3(x)$ (the cubic equation defining v_b), we have $\frac{c_p(c_a + (\delta - 1)p)(\pi_i\alpha_i - 2A_1) + A1(c_a + (\delta - 1)p)^2 + c_p^2(A_1 + \pi_a\alpha_a - \pi_i\alpha_i)}{\pi_a\alpha_a} + \sum_{k=1}^{\infty} e_k \left(\frac{1}{\pi_s\alpha_s}\right)^k = 0$ for some coefficients e_k. Solving for A_1 gives $A_1 = -\frac{c_p(\pi_i\alpha_i(c_a + (\delta - 1)p) + c_p(\pi_a\alpha_a - \pi_i\alpha_i))}{(c_a - c_p + (\delta - 1)p)^2}$. Successively iterating in this way, we can solve for the coefficients in the Taylor series for v_b, giving

$$v_b(p, \delta) = \frac{-c_a + c_p - \delta p + p}{\pi_a\alpha_a} - \frac{(c_p(\pi_i\alpha_i(c_a + (\delta - 1)p) + c_p(\pi_a\alpha_a - \pi_i\alpha_i)))}{\pi_s\alpha_s(c_a - c_p + (\delta - 1)p)^2} +$$

$$\frac{(c_p\pi_a\alpha_a(c_a + (\delta - 1)p)(\pi_i\alpha_i(c_a + (\delta - 1)p) + c_p(\pi_a\alpha_a - \pi_i\alpha_i))(\pi_i\alpha_i(c_a + (\delta - 1)p) + c_p(2\pi_a\alpha_a - \pi_i\alpha_i)))}{(\pi_s\alpha_s)^2(c_a - c_p + (\delta - 1)p)^5} +$$

$$\sum_{k=3}^{\infty} d_k \left(\frac{1}{\pi_s\alpha_s}\right)^k \tag{A.69}$$

Substituting (A.69) into the profit function (A.68), differentiating with respect to p for the first-order condition, and then substituting in $p = \sum_{k=0}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k$ to iteratively solve for the

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coefficients a_k as done above for $v_b(p,\delta)$, we get

$$p^*(\delta) = -\frac{c_a(\pi_a\alpha_a + \delta - 1) + (\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_p(\delta - 1))}{2\left(2\delta(\pi_a\alpha_a - 1) - \pi_a\alpha_a + \delta^2 + 1\right)} +$$

$$\left(\pi_s\alpha_s\left(-\pi_a\alpha_a(\delta - 1)(\pi_a\alpha_a - 1) + c_a\left(\delta(3\pi_a\alpha_a - 2) - \pi_a\alpha_a + \delta^2 + 1\right) - \right.\right.$$

$$\left.\left. c_p\left(\pi_a\alpha_a\delta^2 + 2\pi_a\alpha_a\delta - \pi_a\alpha_a + \delta^2 - 2\delta + 1\right)\right)^3\right)^{-1}\left(\left(2\pi_a\alpha_a c_p(\delta - 1)(\pi_a\alpha_a - 1)\left(2\delta(\pi_a\alpha_a - 1) - \pi_a\alpha_a + \delta^2 + 1\right)\right.\right.$$

$$\left(\pi_i\alpha_i c_a^2\left(\delta(3\pi_a\alpha_a - 2) - \pi_a\alpha_a + \delta^2 + 1\right) + c_a\left(c_p\left((\pi_a\alpha_a)^2(5\delta - 3) + \pi_a\alpha_a\left(\delta^2(3 - \pi_i\alpha_i) - \delta(5\pi_i\alpha_i + 6)+\right.\right.\right.$$

$$\left.\left. 2\pi_i\alpha_i + 3\right) - 2\pi_i\alpha_i(\delta - 1)^2\right) - \pi_a\alpha_a\pi_i\alpha_i(\delta - 1)(\pi_a\alpha_a - 1)\right) + c_p\left(\pi_a\alpha_a(\delta - 1)(\pi_a\alpha_a - 1)(\pi_a\alpha_a + \pi_i\alpha_i)+\right.$$

$$c_p\left((\pi_a\alpha_a)^2\left(\delta^2 - 6\delta + 3\right) + \pi_a\alpha_a\left(\delta^2(\pi_i\alpha_i - 3) + 2\delta(\pi_i\alpha_i + 3) - \pi_i\alpha_i - 3\right) + \right.$$

$$\left.\left.\left.\left.\left. \pi_i\alpha_i(\delta - 1)^2\right)\right)\right)\right)\right) + \sum_{k=2}^{\infty} a_k\left(\frac{1}{\pi_s\alpha_s}\right)^k. \quad (A.70)$$

Substituting (A.70) into the profit function (A.68), differentiating with respect to δ for the first-order condition, and then substituting in $\delta = \sum_{k=0}^{\infty} b_k\left(\frac{1}{\pi_s\alpha_s}\right)^k$ to iteratively solve for the coefficients b_k, we get

$$\delta_{III}^* = \frac{1 - \pi_a\alpha_a - c_a}{1 - c_p} - \left(4c_p\pi_a\alpha_a(\pi_a\alpha_a + c_a - 1)\left(\pi_i\alpha_i c_a^2 + c_a(-\pi_a\alpha_a\pi_i\alpha_i + 3\pi_a\alpha_a c_p - 2\pi_i\alpha_i c_p)+\right.\right.$$

$$\left.\left. c_p(\pi_a\alpha_a(\pi_a\alpha_a + \pi_i\alpha_i) + c_p(\pi_i\alpha_i - 3\pi_a\alpha_a))\right)\right)\left(\pi_s\alpha_s(c_p - 1)^2(\pi_a\alpha_a - c_a + c_p)^3\right)^{-1} + \sum_{k=2}^{\infty} b_k\left(\frac{1}{\pi_s\alpha_s}\right)^k$$

$$(A.71)$$

for some coefficients b_k.

Substituting this into (A.70), we have that

$$p_{III}^* = \frac{1 - c_p}{2} +$$

$$\frac{2\pi_a\alpha_a c_p\left(\pi_i\alpha_i c_a^2 + c_a(-\pi_a\alpha_a\pi_i\alpha_i + 3\pi_a\alpha_a c_p - 2c_p\pi_i\alpha_i) + c_p(\pi_a\alpha_a(\pi_a\alpha_a + \pi_i\alpha_i) + c_p(\pi_i\alpha_i - 3\pi_a\alpha_a))\right)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a - c_p)^3} +$$

$$\sum_{k=2}^{\infty} a_k\left(\frac{1}{\pi_s\alpha_s}\right)^k \quad (A.72)$$

The second-order conditions are satisfied, and the profit at this maximizer is given as

$$\Pi_{III}^* = \frac{1}{4}\left(\frac{c_a^2}{1 - \pi_a\alpha_a} + \frac{(c_a - c_p)^2}{\pi_a\alpha_a} - 2c_p + 1\right) +$$

$$\frac{c_p(\pi_a\alpha_a + c_a - c_p)(\pi_i\alpha_i(c_a - \pi_a\alpha_a) + c_p(2\pi_a\alpha_a - \pi_i\alpha_i))}{\pi_s\alpha_s(\pi_a\alpha_a - c_a + c_p)^2} + \sum_{k=2}^{\infty} c_k\left(\frac{1}{\pi_s\alpha_s}\right)^k \quad (A.73)$$

Next, suppose $0 < v_b < 1$ is induced. By part (IV) of Lemma A.4, we obtain that $v_b = \frac{1}{2} +$

$\frac{-1+\pi_i\alpha_i+\sqrt{(1-\pi_s\alpha_s-\pi_i\alpha_i)^2+4p\pi_s\alpha_s}}{2\pi_s\alpha_s}$. The profit function in this case is

$$\Pi_{IV}(p,\delta) = p(1 - v_b(p,\delta)). \tag{A.74}$$

For brevity of exposition, we will just quickly give the optimal prices and profits after writing the profit function for the remaining cases. The derivation is the same as these previous cases.

The optimal price is given as

$$p_{IV}^* = \frac{1}{9\pi_s\alpha_s}\Bigg(-(\pi_i\alpha_i)^2 + 2\pi_i\alpha_i(1 - 2\pi_s\alpha_s) + \pi_s\alpha_s(4 - \pi_s\alpha_s) - 1 +$$

$$\sqrt{(-\pi_i\alpha_i + \pi_s\alpha_s + 1)^2 \left(\pi_s\alpha_s(\pi_i\alpha_i - 1) + (\pi_i\alpha_i - 1)^2 + (\pi_s\alpha_s)^2\right)}\Bigg) \tag{A.75}$$

The discount δ can be any δ high enough to satisfy the conditions of part (IV) of Lemma A.4 are met under optimal pricing. The profit induced by the optimal price is given as

$$\Pi_{IV}^* = \frac{1}{54(\pi_s\alpha_s)^2}\Bigg(\bigg((\pi_i\alpha_i)^2 - \sqrt{(-\pi_i\alpha_i + \pi_s\alpha_s + 1)^2\left(\pi_s\alpha_s(\pi_i\alpha_i - 1) + (\pi_i\alpha_i - 1)^2 + (\pi_s\alpha_s)^2\right)} +$$

$$2\pi_i\alpha_i(2\pi_s\alpha_s - 1) + \pi_s\alpha_s(\pi_s\alpha_s - 4) + 1\bigg)\bigg(3\pi_i\alpha_i - 3\pi_s\alpha_s - 3 + \bigg(5(\pi_i\alpha_i)^2 +$$

$$4\sqrt{(-\pi_i\alpha_i + \pi_s\alpha_s + 1)^2\left(\pi_s\alpha_s(\pi_i\alpha_i - 1) + (\pi_i\alpha_i - 1)^2 + (\pi_s\alpha_s)^2\right)} + 2\pi_i\alpha_i(\pi_s\alpha_s - 5) + \pi_s\alpha_s(5\pi_s\alpha_s - 2) + 5\bigg)^{1/2}\bigg)\bigg) \tag{A.76}$$

To compare this with the asymptotic profit expressions for the other cases, it will be helpful to also represent the above profit as a Taylor series. There exists $\alpha_7 > 0$ such that for $\alpha_s > \alpha_7$, the profit above can be written as

$$\Pi_{IV}^* = \frac{1}{4\pi_s\alpha_s} - \frac{1}{8(\pi_s\alpha_s)^2} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k \tag{A.77}$$

Next, suppose $0 < v_b < v_a < 1$ is induced. By part (V) of Lemma A.4, we obtain that v_b is the most positive root of the cubic $f_4(x) \triangleq (1 - \pi_a\alpha_a)\pi_s\alpha_s x^3 + ((1 - \pi_a\alpha_a)(1 - \pi_i\alpha_i) - c_a\pi_s\alpha_s - \delta p\pi_s\alpha_s)x^2 + (p(-1 + \pi_a\alpha_a) + p(-1 + \pi_i\alpha_i))x + p^2$ and $v_a = \frac{(c_a - (1-\delta)p)v_b}{v_b(1 - \pi_a\alpha_a) - p}$. The profit function in this case is

$$\Pi_V(p,\delta) = \delta p(1 - v_a(p,\delta)) + p(v_a(p,\delta) - v_b(p,\delta)). \tag{A.78}$$

The optimal price, if interior, is given as

$$p_V^* = \frac{(1 - \pi_i\alpha_i)(-\pi_a\alpha_a + c_a + 1)}{4(1 - \pi_a\alpha_a)} - \frac{\left((\pi_i\alpha_i - 1)^2\left((\pi_a\alpha_a - 1)^2 + c_a(3\pi_a\alpha_a - 2\pi_i\alpha_i - 1)\right)\right)}{16(c_a\pi_s\alpha_s(\pi_a\alpha_a - 1))} + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.79}$$

The optimal discount, if interior, is given as

$$\delta_V^* = \frac{2(1 - \pi_a\alpha_a)(1 - \pi_a\alpha_a - c_a)}{(1 - \pi_i\alpha_i)(1 - \pi_a\alpha_a + c_a)} +$$

$$\frac{(1 - \pi_a\alpha_a)(\pi_a\alpha_a + c_a - 1)\left((\pi_a\alpha_a - 1)^2 + c_a(3\pi_a\alpha_a - 2\pi_i\alpha_i - 1)\right)}{2c_a\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2} + \sum_{k=2}^{\infty} b_k\left(\frac{1}{\pi_s\alpha_s}\right)^k. \quad \text{(A.80)}$$

The profit induced by the interior maximizer in this case is given by

$$\Pi_V^* = \frac{(\pi_a\alpha_a + c_a - 1)^2}{4(1 - \pi_a\alpha_a)} + \frac{\left(c_a(1 - \pi_i\alpha_i)^2\right)}{4(\pi_s\alpha_s(1 - \pi_a\alpha_a))} + \sum_{k=2}^{\infty} c_k\left(\frac{1}{\pi_s\alpha_s}\right)^k. \quad \text{(A.81)}$$

Next, suppose $0 < v_b < v_a < v_p < 1$ is induced. By part (VI) of Lemma A.4, we obtain that v_b is the most positive root of $f_4(x)$, $v_a = \frac{(c_a - (1-\delta)p)v_b}{v_b(1 - \pi_a\alpha_a) - p}$, and $v_p = \frac{(1-\delta)p + c_p - c_a}{\pi_a\alpha_a}$. The profit function in this case is

$$\Pi_{VI}(p, \delta) = \delta p(v_p(p, \delta) - v_a(p, \delta)) + p((1 - v_p(p, \delta)) + (v_a(p, \delta) - v_b(p, \delta))). \quad \text{(A.82)}$$

The optimal price, if interior, is given as

$$p_{VI}^* = \frac{1 - c_p}{2} + \frac{c_a(c_a - c_p\pi_a\alpha_a + c_p)(c_a(1 - \pi_i\alpha_i) + (1 - \pi_a\alpha_a)(-\pi_i\alpha_i + 2c_p - 1))}{\pi_s\alpha_s(1 + c_a - \pi_a\alpha_a)^3} + \sum_{k=2}^{\infty} a_k\left(\frac{1}{\pi_s\alpha_s}\right)^k.$$
$$\text{(A.83)}$$

The optimal discount, if interior, is given as

$$\delta_{VI}^* = \frac{1 - c_a - \pi_a\alpha_a}{1 - c_p} -$$

$$\frac{2c_a\left(c_a^2 + (\pi_a\alpha_a - 1)(\pi_a\alpha_a + c_p^2 - 2c_p\pi_a\alpha_a)\right)(c_a(\pi_i\alpha_i - 1) + (\pi_a\alpha_a - 1)(-\pi_i\alpha_i + 2c_p - 1))}{(c_p - 1)^2\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^3} + \sum_{k=2}^{\infty} b_k\left(\frac{1}{\pi_s\alpha_s}\right)^k.$$
$$\text{(A.84)}$$

The profit induced by the interior maximizer in this case is given by

$$\Pi_{VI}^* = \frac{1}{4}\left(\frac{c_a^2}{1 - \pi_a\alpha_a} + \frac{(c_a - c_p)^2}{\pi_a\alpha_a} - 2c_p + 1\right) +$$

$$\frac{c_a(1 - c_p)(c_a(1 - \pi_i\alpha_i) + (1 - \pi_a\alpha_a)(c_p - \pi_i\alpha_i))}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2} + \sum_{k=2}^{\infty} c_k\left(\frac{1}{\pi_s\alpha_s}\right)^k. \quad \text{(A.85)}$$

Lastly, suppose $0 < v_b < v_p < 1$ is induced. By part (VII) of Lemma A.4, we obtain that v_b is the most positive root of $f_5(x) \triangleq \pi_s\alpha_s x^3 + (1 - \pi_i\alpha_i - (c_p + p)\pi_s\alpha_s)x^2 - p(2 - \pi_i\alpha_i)x + p^2$ and $v_p = \frac{c_p v_b}{v_b - p}$. The profit function in this case is

$$\Pi_{VII}(p, \delta) = p(1 - v_b(p, \delta)). \quad \text{(A.86)}$$

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The optimal price, if interior, is given as

$$p^*_{VII} = \frac{1 - c_p}{2} - \frac{\left(2c_p^2(\pi_i\alpha_i(c_p + 1) - 3c_p + 1)\right)}{(c_p + 1)^3\pi_s\alpha_s} + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.87}$$

The discount δ can be any δ high enough to satisfy the conditions of part (VII) of Lemma A.4 are met under optimal pricing. The profit induced by the interior maximizer in this case is given by

$$\Pi^*_{VII} = \frac{1}{4}(1 - c_p)^2 - \frac{c_p(1 - c_p)(\pi_i\alpha_i(c_p + 1) - 2c_p)}{(c_p + 1)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.88}$$

To find the conditions under which the interior maximizer of each case indeed induces the correct market structure, we follow the same steps as in Lemma 4 by finding the conditions under which the interior maximizer satisfies the conditions of the cases in Lemma A.4. For brevity, we omit the algebra.

To have $0 < v_b < v_a < v_p < 1$ be induced by the maximizing prices given by (A.83) and (A.84), the conditions are $|c_p - \pi_a\alpha_a| < c_a < c_p(1 - \pi_a\alpha_a)$ and $\pi_i\alpha_i < \frac{(c_a + c_p(1 - \pi_a\alpha_a))}{1 + c_a - \pi_a\alpha_a}$. Note that $\frac{(c_a + c_p(1 - \pi_a\alpha_a))}{1 + c_a - \pi_a\alpha_a} > \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}$ from $0 < c_p < 1$, $0 < c_a < 1$, and $0 < \pi_a\alpha_a < 1$, so that the condition of the lemma $\pi_i\alpha_i < \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}$ implies $\pi_i\alpha_i < \frac{(c_a + c_p(1 - \pi_a\alpha_a))}{1 + c_a - \pi_a\alpha_a}$.

To have $0 < v_a < v_b < v_p < 1$ be induced by the maximizing prices given by (A.72) and (A.71), the conditions are $c_a < \min[c_p(1 - \pi_a\alpha_a), \pi_a\alpha_a - c_p]$ and $\pi_i\alpha_i < \frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a}$. Note that $\frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a} > \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}$ from $0 < \pi_a\alpha_a < 1$ and $0 < c_a < c_p(1 - \pi_a\alpha_a)$, so that the condition of the lemma $\pi_i\alpha_i < \frac{c_p\pi_a\alpha_a}{1 + c_p - c_a}$ implies $\pi_i\alpha_i < \frac{2c_p\pi_a\alpha_a}{c_p - c_a + \pi_a\alpha_a}$.

To have $0 < v_b < v_p < 1$ be induced by the maximizing price given by (A.87), the conditions are $\pi_i\alpha_i < \frac{2c_p}{1 + c_p}$ and $\delta \geq \frac{-2c_a + (1 + c_p)(1 - \pi_a\alpha_a)}{1 - c_p}$. Note that the condition $\pi_i\alpha_i < \frac{2c_p}{1 + c_p}$ holds since one of the conditions of this lemma is $\pi_i\alpha_i < \frac{c_p}{1 + c_p}$. Then given any parameters in the parameter space satisfying the lemma, this case $0 < v_b < v_p < 1$ can always be induced with any δ large enough to satisfy these conditions.

Now we compare the maximizing profits of each case to establish the lemma. By comparing (A.73) and (A.85) with (A.59), (A.67), (A.81), (A.77), and (A.88), it follows that there exists $\alpha_8 > 0$ such that if $\alpha_s > \alpha_8$, if either $0 < v_b < v_a < v_p < 1$ or $0 < v_a < v_b < v_p < 1$ can be induced by their maximizing prices, then they will because they dominate the profits of the other cases. Furthermore, since (A.73) can only be achieved when $c_a < \min[c_p(1 - \pi_a\alpha_a), \pi_a\alpha_a - c_p]$ and (A.85) can only be achieved when $|c_p - \pi_a\alpha_a| < c_a < c_p(1 - \pi_a\alpha_a)$ (which doesn't overlap with the region over which (A.73) can be achieved), it follows that p^* and δ^* are set so that

1. if $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing,

2. if $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing.

When $c_a = c_p(1 - \pi_a\alpha_a)$, then the maximal profit when inducing $0 < v_b < v_p < 1$ equals the maximal profits when inducing either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$. Furthermore, for $c_a \geq c_p(1 - \pi_a\alpha_a)$, by comparing (A.88) with (A.59), (A.67), (A.81), and (A.77), it follows that

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there exists $\alpha_9 > 0$ such that if $\alpha_s > \alpha_9$, then the profit of $0 < v_b < v_p < 1$ will dominate the other cases. Also, $\delta = 1$ can be set to induce this case since $\delta = 1$ satisfies $\delta \geq \frac{-2c_a + (1+c_p)(1-\pi_a\alpha_a)}{1-c_p}$ when $c_a \geq c_p(1 - \pi_a\alpha_a)$. Altogether, when $\alpha_s > \tilde{\alpha}_2 \triangleq \max[\alpha_8, \alpha_9]$ and if $\pi_i\alpha_i < \min\left[\frac{c_p\pi_a\alpha_a}{1+c_p-c_a}, \frac{c_p}{1+c_p}\right]$, then p^* and δ^* are set so that

1. if $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing,

2. if $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and if

3. if $c_a > c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

$\square$

Proof of Proposition 2: We focus on the region in which all segments are represented under optimal pricing in the base case. Specifically, for $\alpha_s > \tilde{\alpha}_1$, by Lemma 4, we have that p^* is set so that if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1 - \pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing. By Lemma 5, for $\alpha_s > \tilde{\alpha}_2$, when patching rights are priced under the same parameter region, there are two cases: either $0 < v_a < v_b < v_p < 1$ is induced or $0 < v_b < v_a < v_p < 1$ is induced. Specifically, p^* and δ^* are set so that

(i) if $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing, and

(ii) if $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing.

In either case, since $c_p(1 - \pi_a\alpha_a) > 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$ using the assumptions that $0 < c_p < 1$ and $0 < \pi_a\alpha_a < 1$, we have that $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$ is a subset of the union of the regions $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$ and $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$. Moreover, the intersection of $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a}$ with either $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$ or $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$ is non-empty.

In the first case, the induced profit under optimal pricing in the status quo case when patching rights aren't priced is given by (A.18), and induced profit when patching rights are priced is given by (A.73). The fractional increase in profit is given by

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2} + \left((\pi_a\alpha_a - c_a + c_p)^2 M - 4\pi_a\alpha_a(1 - \pi_a\alpha_a)(1 - \pi_a\alpha_a - c_a) \right)$$

$$(-\pi_a\alpha_a + c_a + c_p)\left(\pi_a\alpha_a((c_a + 2)c_p + c_a) + c_a(1 - c_p)(c_p - c_a) - 2c_p(\pi_a\alpha_a)^2\right)(c_a - c_p(1 - \pi_a\alpha_a)) +$$

$$4\pi_i\alpha_i(\pi_a\alpha_a - 1)(-\pi_a\alpha_a + c_a + 1)(-\pi_a\alpha_a + c_a - c_p)\left((\pi_a\alpha_a)^3(\pi_a\alpha_a)^2(c_a + c_p + 1) - \pi_a\alpha_a(c_a - 1)(c_a + c_p) +\right.$$

$$\left. c_a(c_a - c_p)^2\right)(c_a + c_p(\pi_a\alpha_a - 1)) \right) \left(\pi_a\alpha_a\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2(\pi_a\alpha_a + c_a - 1)^3(\pi_a\alpha_a - c_a + c_p)^2 \right)^{-1} + K_h\,,$$

$$(A.89)$$

A.33

where

$$M = 4c_a(\pi_a\alpha_a-1)(\pi_a\alpha_a+c_a-c_p)\left((\pi_a\alpha_a + c_a)\left(\pi_a\alpha_a(2 - \pi_a\alpha_a) + c_a(\pi_a\alpha_a - 2) + 2c_p(\pi_a\alpha_a - 1)^2\right) - \pi_a\alpha_a\right) \quad (A.90)$$

and K_h is a term of order $O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$.

Moreover, the reduction in the size of the unpatched population when patching rights are priced is given by

$$\hat{u}(\sigma^*|SQ)-\hat{u}(\sigma^*|PPR) = \frac{\left(c_a^2(\pi_a\alpha_a - 2) + c_a(\pi_a\alpha_a + c_p(2 - 3\pi_a\alpha_a)) + \pi_a\alpha_a(\pi_a\alpha_a - 1)(c_p - \pi_a\alpha_a)\right)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)(\pi_a\alpha_a - c_a + c_p)} +$$
$$\sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k, \quad (A.91)$$

which is strictly positive when $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)\right]$.

Similarly, in the second case, the induced profit under optimal pricing in the status quo case when patching rights aren't priced is given by (A.18), and induced profit when patching rights are priced is given by (A.85). The fractional increase in profit is given by

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1 - \pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1 - c_a - \pi_a\alpha_a)^2} + \Bigg(M+(-\pi_a\alpha_a+c_a+1)\big(4\pi_i\alpha_i c_a(\pi_a\alpha_a-1)(\pi_a\alpha_a+c_a-c_p)$$
$$(c_a + c_p(\pi_a\alpha_a - 1))\big)\Bigg)\left(\pi_a\alpha_a\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2(\pi_a\alpha_a + c_a - 1)^3\right)^{-1}K_k, \quad (A.92)$$

where K_k is a term of order $O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$.

Moreover, the reduction in the size of the unpatched population when patching rights are priced is given by

$$\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR) = \frac{(1 - \pi_a\alpha_a)(\pi_a\alpha_a + c_a - c_p)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)} + \sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k, \quad (A.93)$$

which is strictly positive when $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$.

In both cases, pricing patching rights increases the vendor's profit and reduces the equilibrium size of the unpatched population as compared to when patching rights aren't priced. $\square$

Proof of Corollary 1: Differentiating (A.89) and (A.92) with respect to α_i, we have that in either case, the profit difference between pricing patching rights and the status quo case decreases in α_i.

In the first case when $0 < v_a < v_b < v_p < 1$ is induced under pricing patching rights, the derivative of the profit difference with respect to α_i is given as

$$\frac{d}{d\alpha_i}\left(\Pi_P^* - \Pi_{SQ}^*\right) = \frac{\pi_i\left(-c_a^3 + c_a^2(2c_p + 1) + c_a\left(\pi_a\alpha_a(\pi_a\alpha_a - 1) - c_p^2 + c_p\pi_a\alpha_a\right) + c_p(\pi_a\alpha_a - 1)(c_p - \pi_a\alpha_a)\right)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)(-\pi_a\alpha_a + c_a - c_p)} +$$
$$\sum_{k=2}^{\infty} c_k\left(\frac{1}{\pi_s\alpha_s}\right)^k, \quad (A.94)$$

A.34

which is negative when $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$.

In the second case when $0 < v_b < v_a < v_p < 1$ is induced under pricing patching rights, the derivative of the profit difference with respect to α_i is given as

$$\frac{d}{d\alpha_i}\left(\Pi_P^* - \Pi_{SQ}^*\right) = -\frac{c_a\pi_i(\pi_a\alpha_a + c_a - c_p)}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)} + \sum_{k=2}^{\infty} c_k\left(\frac{1}{\pi_s\alpha_s}\right)^k, \tag{A.95}$$

which is negative when $|\pi_a\alpha_a - c_p| < c_a < c_p(1 - \pi_a\alpha_a)$. $\square$

Proof of Corollary 2: Following the proof of Proposition 2, the reduction in the size of the unpatched population under the conditions of the corollary is given either by (A.91) or (A.93), depending on the parameters (with the conditions also given in the proof of Proposition 2).

In the first case, $\frac{d}{dc_p}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] = \frac{2\pi_a\alpha_a(c_a - \pi_a\alpha_a)}{(c_a - c_p - \pi_a\alpha_a)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k$. Using $c_a < \pi_a\alpha_a - c_p$ (one of the conditions for this case, from Lemma 5), there exists $\hat{\alpha}_s$ such that $\alpha_s > \hat{\alpha}_s$ implies that $\frac{d}{dc_p}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] < 0$.

Also,

$$\frac{d}{d(\pi_a\alpha_a)}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] = \frac{1}{\pi_s\alpha_s}\left(1 + (2(c_p - c_a)(c_a^3 - (1 + 2c_a(1 + c_a))c_p) + 4(c_a - c_p)\right.$$

$$(c_a^2 - (1 + c_a)c_p)\pi_a\alpha_a - 2(c_a^2 - c_ac_p + c_p^2)(\pi_a\alpha_a)^2)((1 + c_a - \pi_a\alpha_a)^2(-c_a + c_p + \pi_a\alpha_a)^2)^{-1}) + \sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.96}$$

There exists $\hat{\alpha}_s$ such that $\alpha_s > \hat{\alpha}_s$ implies that this is strictly positive, since $c_a < \pi_a\alpha_a - c_p$ in this parameter region and $\pi_a\alpha_a + c_a < 1$ from our initial model assumptions.

In the second case, $\frac{d}{dc_p}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] = \frac{-(1 - \pi_a\alpha_a)}{(1 + c_a - \pi_a\alpha_a)\pi_s\alpha_s} + \sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k$. There exists $\hat{\alpha}_s$ such that $\alpha_s > \hat{\alpha}_s$ implies that $\frac{d}{dc_p}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)]$ is strictly negative.

Also, $\frac{d}{d(\pi_a\alpha_a)}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] = \frac{-c_a^2 + c_a(1 + c_p - 2\pi_a\alpha_a) + (1 - \pi_a\alpha_a)^2}{(1 + c_a - \pi_a\alpha_a)^2\pi_s\alpha_s} + \sum_{k=2}^{\infty} d_k\left(\frac{1}{\pi_s\alpha_s}\right)^k$.

Note that $-c_a^2 + c_a(1 + c_p - 2\pi_a\alpha_a) + (1 - \pi_a\alpha_a)^2 > 0$, using $c_a < c_p(1 - \pi_a\alpha_a)$ and $c_a > \pi_a\alpha_a - c_p$ (from Lemma 5). Therefore, for sufficiently large $\pi_s\alpha_s$, $\frac{d}{d(\pi_a\alpha_a)}[\hat{u}(\sigma^*|SQ) - \hat{u}(\sigma^*|PPR)] > 0$. $\square$

Proof of Proposition 3: Under the conditions of the Proposition 3, when patching rights are priced, there are two cases in equilibrium by Lemma 5: either $0 < v_a < v_b < v_p < 1$ is induced or $0 < v_b < v_a < v_p < 1$ is induced. In either case, we show that the optimal discount $\delta^* < 1$. It follows that it suffices to show that the price of the automated patching option can be higher when patching rights are priced than compared to the status quo case for low c_a.

In the first region when $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0 < v_a < v_b < v_p < 1$ under optimal pricing. The discount when pricing patching rights is given in (A.71). Under the conditions $c_a < \min[\pi_a\alpha_a - c_p, c_p(1 - \pi_a\alpha_a)]$, there exists $\tilde{\alpha}_3 > 0$ such that $\alpha_s > \tilde{\alpha}_3$ implies the expression for δ^* given in (A.71) is bounded above by 1. To prove the lemma for this case, it suffices to show that the price of the automated patching option can be greater than the common status quo price across both options.

The price of the automated patching option when patching rights aren't priced is given in (A.17). Using (A.72) and (A.71), when patching rights are priced, the automated patching option has price

$$\delta^* p^* = \frac{1}{2}\left(1 - c_a - \pi_a\alpha_a\right) + \sum_{k=2}^{\infty} a_k \left(\frac{1}{\pi_s\alpha_s}\right)^k.$$ (A.97)

Comparing (A.97) with (A.17), the price of the automated patching option when patching rights are priced is greater than the common price in the base case when $c_a < \frac{\pi_i\alpha_i(2\pi_a\alpha_a-1)-1}{2\pi_a\alpha_a+\pi_i\alpha_i-3} - \pi_a\alpha_a$. The intersection of this with the parameter region of this case is non-empty when $\pi_a\alpha_a < \frac{1+\pi_i\alpha_i}{2}$.

The argument for the second case (in which $0 < v_b < v_a < v_p < 1$ is induced in equilibrium) is similar and is omitted for brevity. We find that $\delta^* p^*$ is greater than the price of the status quo case if $c_a < \frac{(1-\pi_a\alpha_a)(\pi_i\alpha_i-2c_p+1)}{-4\pi_a\alpha_a-\pi_i\alpha_i+5}$. The intersection of this with the parameter region of this case, namely with the condition $c_a > \pi_a\alpha_a - c_p$, is non-empty also when $\pi_a\alpha_a < \frac{1+\pi_i\alpha_i}{2}$. $\square$

Proof of Proposition 4: Using Lemma 4, the status quo pricing induces $0 < v_b < v_a < v_p < 1$ market structure. Using Lemma 5, the vendor induces either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$ under PPR. Using the definition of social welfare in (35), the social welfare under status quo pricing is given by

$$W_{SQ} = \int_{v_b(p^*)}^{1} v\,dv - \left(\int_{v_a(p^*)}^{v_p(p^*)} c_a + \pi_a\alpha_a v\,dv + \int_{v_b(p^*)}^{v_a(p^*)} ((v_a(p^*) - v_b(p^*))\pi_s\alpha_s + \pi_i\alpha_i)v\,dv + c_p(1 - v_p(p^*))\right),$$ (A.98)

where p^* is the equilibrium price in the status quo case, given in (A.17). Using its asymptotic expansion, this can be written as

$$W_{SQ} = \frac{1}{8}\left(\pi_a\alpha_a - \frac{3c_a^2}{\pi_a\alpha_a - 1} + \frac{4(c_a - c_p)^2}{\pi_a\alpha_a} + 2c_a - 8c_p + 3\right) + \sum_{k=1}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k.$$ (A.99)

When pricing patching rights, one of the two cases will arise. In the first case, the social welfare is given as

$$W_P = \int_{v_a(p^*,\delta^*)}^{1} v\,dv - \left(\int_{v_a(p^*,\delta^*)}^{v_b(p^*,\delta^*)} c_a + \pi_a\alpha_a v\,dv + \right.$$
$$\left.\int_{v_b(p^*,\delta^*)}^{v_p(p^*,\delta^*)} ((v_p(p^*,\delta^*) - v_b(p^*,\delta^*))\pi_s\alpha_s + \pi_i\alpha_i)v\,dv + c_p(1 - v_p(p^*,\delta^*))\right),$$ (A.100)

where δ^* and p^* are given in (A.71) and (A.72) respectively. In the second case, the social welfare is given as

$$W_P = \int_{v_b(p^*,\delta^*)}^{1} v\,dv - \left(\int_{v_a(p^*,\delta^*)}^{v_p(p^*,\delta^*)} c_a + \pi_a\alpha_a v\,dv + \right.$$
$$\left.\int_{v_b(p^*,\delta^*)}^{v_a(p^*,\delta^*)} ((v_a(p^*,\delta^*) - v_b(p^*,\delta^*))\pi_s\alpha_s + \pi_i\alpha_i)v\,dv + c_p(1 - v_p(p^*,\delta^*))\right),$$ (A.101)

where δ^* and p^* are given in (A.84) and (A.83) respectively.

In both cases, the asymptotic expression for the equilibrium welfare is given as

$$W_P = \frac{3}{8}\left(\frac{c_a^2}{1-\pi_a\alpha_a} + \frac{(c_a-c_p)^2}{\pi_a\alpha_a} - 2c_p + 1\right) + \sum_{k=1}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.102}$$

Comparing (A.99) and (A.102) reveals that pricing patching rights in this region of the parameter space hurts welfare.

We further characterize which losses drive this result. We define the total attack-related losses under status quo pricing with

$$SL_{SQ} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP)|SQ\}} \left(\pi_s\alpha_s u(\sigma^*|SQ) + \pi_i\alpha_i\right) v\, dv, \tag{A.103}$$

the total costs associated with automated patching under status quo pricing with

$$AL_{SQ} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,AP)|SQ\}} c_a + \pi_a\alpha_a v\, dv, \tag{A.104}$$

and the total costs associated with standard patching under status quo pricing with

$$PL_{SQ} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)|SQ\}} c_p\, dv. \tag{A.105}$$

Specifically, the loss measures when $0 < v_a < v_b < v_p < 1$ is induced in equilibrium under status quo pricing are given as follows.

$$SL_{SQ} = -\Bigg(\big((\pi_a\alpha_a(\pi_a\alpha_a - 1) + c_a(\pi_a\alpha_a - 2))((\pi_a\alpha_a - 1)(\pi_a\alpha_a - \pi_i\alpha_i) + c_a(\pi_a\alpha_a+$$

$$\pi_i\alpha_i - 2)))\Bigg)\Big(2(\pi_s\alpha_s(\pi_a\alpha_a - 1)(-\pi_a\alpha_a + c_a + 1))\Big)^{-1} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k, \tag{A.106}$$

$$AL_{SQ} = \Bigg((\pi_a\alpha_a(\pi_a\alpha_a - 1) + c_a(\pi_a\alpha_a - 2))\big((\pi_a\alpha_a - 1)^3(\pi_a\alpha_a - \pi_i\alpha_i) + c_a^3(3\pi_a\alpha_a + \pi_i\alpha_i - 4)+$$

$$c_a^2(\pi_a\alpha_a-1)(3\pi_a\alpha_a-\pi_i\alpha_i-2)+c_a(\pi_a\alpha_a-1)^2(\pi_a\alpha_a+\pi_i\alpha_i-2))\Bigg)\Big(2\pi_s\alpha_s(\pi_a\alpha_a-1)(-\pi_a\alpha_a+c_a+1)^3\Big)^{-1} -$$

$$\Bigg(\big((\pi_a\alpha_a)^2 + \pi_a\alpha_a(c_a - 2c_p - 1) - 2c_a + 2c_p\big)\left(c_a(\pi_a\alpha_a - 2) + (\pi_a\alpha_a - 1)(\pi_a\alpha_a+\right.$$

$$\left.2c_p)\right)\Bigg)\Big(8\left(\pi_a\alpha_a(\pi_a\alpha_a - 1)^2\right)\Big)^{-1} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k, \tag{A.107}$$

and

$$PL_{SQ} = \frac{c_p(\pi_a\alpha_a + c_a - c_p)}{\pi_a\alpha_a}\, dv + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \tag{A.108}$$

A.37

Similarly, we define the total attack-related losses under PPR with

$$SL_P \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP)|PPR\}} \left(\pi_s \alpha_s u(\sigma^*|PPR) + \pi_i \alpha_i \right) v \, dv \,, \tag{A.109}$$

the total costs associated with automated patching under PPR with

$$AL_P \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,AP)|PPR\}} c_a + \pi_a \alpha_a v \, dv \,, \tag{A.110}$$

and the total costs associated with standard patching under PPR with

$$PL_P \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)|PPR\}} c_p \, dv \,. \tag{A.111}$$

In the first case, in which $0 < v_a < v_b < v_p < 1$ is induced under PPR, the loss measures in equilibrium are given as follows.

$$SL_P = \frac{c_p(\pi_i \alpha_i (\pi_a \alpha_a - c_a + c_p) - 2\pi_a \alpha_a c_p)}{\pi_s \alpha_s (-\pi_a \alpha_a + c_a - c_p)} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s \alpha_s} \right)^k \,, \tag{A.112}$$

$$AL_P = \left(c_p \big(2c_p \pi_a \alpha_a \left((\pi_a \alpha_a)^2 + c_a^2 - \pi_a \alpha_a (2c_a + c_p) - 3c_a c_p + 2c_p^2 \right) + \pi_i \alpha_i (-\pi_a \alpha_a + c_a - c_p) \right.$$
$$\left. \left((\pi_a \alpha_a)^2 - 2c_a \pi_a \alpha_a + (c_a - c_p)^2 \right) \big) \right) \left(\pi_s \alpha_s (-\pi_a \alpha_a + c_a - c_p)^3 \right) + \frac{c_p(\pi_a \alpha_a + c_a - c_p)}{2\pi_a \alpha_a} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s \alpha_s} \right)^k \,, \tag{A.113}$$

and

$$PL_P = \left(c_p \pi_a \alpha_a (\pi_a \alpha_a + c_a + c_p)(\pi_i \alpha_i (\pi_a \alpha_a - c_a + c_p) + c_p(-3\pi_a \alpha_a - c_a + c_p)) \right) \left(\pi_s \alpha_s (\pi_a \alpha_a - c_a + c_p)^3 \right)^{-1} +$$
$$\frac{(c_a + c_p(\pi_a \alpha_a - 1))(c_a(2\pi_a \alpha_a - 3) + (\pi_a \alpha_a - 1)(2\pi_a \alpha_a + c_p))}{8\pi_a \alpha_a (\pi_a \alpha_a - 1)^2} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s \alpha_s} \right)^k \,. \tag{A.114}$$

Comparing these measures across status quo pricing and pricing patching rights, we have that $AL_P > AL_{SQ}$, $PL_{SQ} > PL_P$, and if $\frac{4c_p^2 \pi_a \alpha_a}{-c_a + c_p + \pi_a \alpha_a} - \frac{(c_a(2-\pi_a \alpha_a)+\pi_a \alpha_a(1-\pi_a \alpha_a))^2}{(1+c_a-\pi_a \alpha_a)(1-\pi_a \alpha_a)} > 0$, then $SL_P > SL_{SQ}$. Otherwise, $SL_P \leq SL_{SQ}$.

In the second case, in which $0 < v_b < v_a < v_p < 1$ is induced under PPR, the loss measures in equilibrium are given as follows.

$$SL_P = -\frac{((c_a - \pi_a \alpha_a c_p + c_p)(-\pi_i \alpha_i(-\pi_a \alpha_a + c_a + 1) + c_a - \pi_a \alpha_a c_p + c_p))}{2(\pi_s \alpha_s (\pi_a \alpha_a - 1)(-\pi_a \alpha_a + c_a + 1))} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s \alpha_s} \right)^k \,, \tag{A.115}$$

A.38

$$AL_P = \left(\left(c_a^4(1 - \pi_i\alpha_i) - c_a^3(\pi_a\alpha_a - 1)(\pi_i\alpha_i(c_p - 2) + c_p) - c_a^2(\pi_a\alpha_a - 1)\left(- \pi_a\alpha_a + \pi_a\alpha_a c_p^2 + \right. \right. \right.$$

$$\pi_i\alpha_i(2\pi_a\alpha_a + c_p(4 - 3\pi_a\alpha_a) - 3) + \pi_a\alpha_a c_p - 4c_p + 3) + c_a(\pi_a\alpha_a - 1)^2(2\pi_a\alpha_a - 2\pi_i\alpha_i + c_p(-\pi_a\alpha_a(\pi_i\alpha_i + 5) +$$

$$\left. \left. 3\pi_i\alpha_i + 4c_p(\pi_a\alpha_a - 1) + 3)) + \pi_a\alpha_a(c_p - 1)(\pi_a\alpha_a - 1)^3(c_p - \pi_i\alpha_i) \right) \left(2\pi_s\alpha_s(\pi_a\alpha_a - 1)(-\pi_a\alpha_a + c_a + 1)^3 \right)^{-1} \right) +$$

$$\frac{(c_a + c_p(\pi_a\alpha_a - 1))(c_a(2\pi_a\alpha_a - 3) + (\pi_a\alpha_a - 1)(2\pi_a\alpha_a + c_p))}{8\pi_a\alpha_a(\pi_a\alpha_a - 1)^2} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s} \right)^k, \quad \text{(A.116)}$$

and

$$PL_P = \frac{c_a c_p(c_a(\pi_i\alpha_i - 1) + (\pi_a\alpha_a - 1)(-\pi_i\alpha_i + 2c_p - 1))}{\pi_s\alpha_s(-\pi_a\alpha_a + c_a + 1)^2} + \frac{c_p(\pi_a\alpha_a + c_a - c_p)}{2\pi_a\alpha_a} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s} \right)^k.$$

$$\text{(A.117)}$$

Comparing these measures across status quo pricing and pricing patching rights, we have that $AL_P > AL_{SQ}$, $PL_{SQ} > PL_P$, and $SL_P < SL_{SQ}$ always holds under the conditions of this case.
□

Proof of Proposition 5: When $1 - \pi_a\alpha_a - (1 - c_p)\sqrt{1 - \pi_a\alpha_a} < c_a < c_p(1 - \pi_a\alpha_a)$, then by Lemmas 4 and 5, status quo pricing induces $0 < v_b < v_a < v_p < 1$ while pricing patching rights induces either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$.

So the welfare expression under PPR is the same as in the proof of Proposition 4. Now however, the welfare under status quo pricing is given as

$$W_{SQ} = \int_{v_b(p^*)}^{1} v\,dv - \left(\int_{v_b(p^*)}^{v_p(p^*)} ((v_p(p^*) - v_b(p^*))\pi_s\alpha_s + \pi_i\alpha_i)v\,dv + c_p(1 - v_p(p^*)) \right), \quad \text{(A.118)}$$

where p^* is the equilibrium price in the status quo case, given in (A.19). Using its asymptotic expansion, this can be written as

$$W_{SQ} = \frac{3}{8}(1 - c_p)^2 + \sum_{k=1}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s} \right)^k. \quad \text{(A.119)}$$

Comparing (A.119) to (A.102) reveals that, for sufficiently high $\pi_s\alpha_s$, pricing patching rights in this region of the parameter space improves welfare. Defining the specific losses for when $0 < v_b < v_p < 1$ arises under status quo pricing, we derive the following loss measures.

$$SL_{SQ} = -\frac{c_p(\pi_i\alpha_i(c_p + 1) - 2c_p)}{(c_p + 1)\pi_s\alpha_s} + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s} \right)^k, \quad \text{(A.120)}$$

$$AL_{SQ} = 0, \quad \text{(A.121)}$$

and

$$PL_{SQ} = \frac{c_p \left(\pi_i\alpha_i(c_p + 1)\left(c_p^2 + 1\right) + 2c_p\left(-2c_p^2 + c_p - 1\right)\right)}{(c_p + 1)^3\pi_s\alpha_s} - \frac{1}{2}(c_p-1)c_p + \sum_{k=2}^{\infty} c_k \left(\frac{1}{\pi_s\alpha_s}\right)^k. \quad \text{(A.122)}$$

Comparing these with their respective measures given in the proof of Proposition 4, we find $SL_P < SL_{SQ}$, $PL_P < PL_{SQ}$, and $AL_P > AL_{SQ}$. $\square$